Unifying Functional Interpretations

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Dagstuhl, January 11, 2004

History

1958. Gödel's Dialectica interpretation

- *Relative consistency of* PA

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1992. Monotone functional interpretation

- *Proof mining*

Goal

Relation between Dialectica interpretation and modified realizability.

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Common framework for all functional interpretations.

Road map

1. Functional Interpretations

2. Parametrised Formula Translation
   - (Standard) Proof Translation for PFT

3. Parametrised Proof Translation

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Logical Framework

- Language of finite types $\mathcal{T}$
  - $\mathbb{N} \in \mathcal{T}$
  - $\rho, \sigma \in \mathcal{T} \quad \Rightarrow \quad \rho \to \sigma \in \mathcal{T}$
- Variable and quantifiers for each finite type $\rho \in \mathcal{T}$
  - $\forall x^\rho A(x)$
  - $\exists x^\rho A(x)$
- HA$^\omega$ $\equiv$ Heyting arithmetic in the language of finite types

Logical Framework

- Constructs:

$A \wedge B$	conjunction
$A \vee B$	disjunction
$A \triangledown B$	classical disjunction
$\neg A$	negation
$\exists x^\rho A$	existential quantifier
$\forall x^\rho A$	universal quantifier

- $A \to B \equiv \neg A \triangledown B$

A Functional Interpretation

- A formula mapping

$$A \quad \mapsto \quad |A|^x_y$$

  - x marks the witness required by A (i.e. $\forall y |A|^t_y$)
  - y marks the refutation of a given witness for A.

- A proof mapping

$$\mathsf{HA}^\omega \vdash A \quad \mapsto \quad \mathsf{HA}^\omega \vdash B,$$

for some B such that $\mathsf{HA}^\omega \vdash B \to \exists x \forall y |A|^x_y$.

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$$\mathsf{HA}^\omega \vdash A \quad \mapsto \quad \mathsf{HA}^\omega \vdash \forall y |A|^t_y,$$

for some term t.

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The Formula Translation

$|A_{\mathsf{at}}| :\equiv A_{\mathsf{at}}$, when A_{at} is atomic.

Assume we have already defined $|A|^x_y$ and $|B|^v_w$, we define

$$
\begin{aligned}
|A \wedge B|^{x,v}_{y,w} &:\equiv |A|^x_y \wedge |B|^v_w, \\
|A \vee B|^{x,v,n}_{y,w} &:\equiv (n = 0 \rightarrow |A|^x_y) \wedge (n \neq 0 \rightarrow |B|^v_w), \\
|A \triangledown B|^{f,g}_{y,w} &:\equiv |A|^{gw}_y \triangledown |B|^{fy}_w, \\
|\forall z A(z)|^f_{y,z} &:\equiv |A(z)|^{fz}_y \\
|\exists z A(z)|^{x,z}_y &:\equiv |A(z)|^x_y.
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- What about $|\neg A|$?

The Witnesses of $\neg A$

Assume A has interpretation $|A|^x_y$

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- Modified Realizability

  $\neg A$ does not ask for witnesses, i.e.

  $|\neg A|_x :\equiv \neg \forall y |A|^x_y$

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- In General:

  Functionals producing "bound" on counter-examples

  $$|\neg A|^f_x :\equiv \neg \forall y \sqsubset fx |A|^x_y \equiv \neg \forall y (y \sqsubset fx \rightarrow |A|^x_y)$$

Parametrised Formula Translation

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|\forall z A(z)|^{f}_{y,z} & :\equiv & |A(z)|^{fz}_y \\
|\exists z A(z)|^{x,z}_{y} & :\equiv & |A(z)|^{x}_y \\
|\neg A|^{f}_{x} & :\equiv & \neg \forall y \sqsubset fx |A|^x_y
\end{array}
$$

Instantiations

- **Modified realizability**

  Choose $\forall x \sqsubset a A(x) :\equiv \forall x A(x)$

  $\mathsf{HA}^\omega \vdash \exists x \forall y |A|^x_y \leftrightarrow \exists x (x \text{ mr } A)$.

- **Dialectica interpretation**

  Choose $\forall x \sqsubset a A(x) :\equiv A(a)$

  $\mathsf{HA}^\omega \vdash \exists x \forall y |A|^x_y \leftrightarrow \exists x \forall y A_D(x, y)$.

- **Diller-Nahm variant**

  Choose $\forall x \sqsubset a A(x) :\equiv \forall x \in a A(x)$

  $\mathsf{HA}^\omega \vdash \exists x \forall y |A|^x_y \leftrightarrow \exists x \forall y A_\wedge(x, y)$.

The $\sqsubset$-Bounded Formulas

Definition.

Those built out of prime formulas via

- conjunction ($A_{\mathrm{b}} \wedge B_{\mathrm{b}}$)
- implication ($A_{\mathrm{b}} \to B_{\mathrm{b}}$)
- bounded quantification ($\forall x \sqsubset t A_{\mathrm{b}}$)

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$\sqsubset$-bounded formulas $\equiv$ $\exists$-free formulas.

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Those built out of prime formulas via

- conjunction ($A_{\rm b} \wedge B_{\rm b}$)
- implication ($A_{\rm b} \to B_{\rm b}$)
- bounded quantification ($\forall x \sqsubset t A_{\rm b} :\equiv A_{\rm b}[t/x]$)

$\sqsubset$-bounded formulas $\equiv$ quantifier-free formulas.

Conditions on Choice of $\forall x \sqsubset a A(x)$

For each $\sqsubset$-bounded formula $A_{\mathrm{b}}(x^\rho)$,

there are terms a_1, a_2, a_3 such that

(A1) $\forall x \sqsubset \mathsf{a}_1\, y\, A_{\mathrm{b}}(x) \rightarrow A_{\mathrm{b}}(y)$,

(A2) $\forall x \sqsubset \mathsf{a}_2\, y_0\, y_1\, A_{\mathrm{b}}(x) \rightarrow \bigwedge_{i=0}^{1} \forall x \sqsubset y_i A_{\mathrm{b}}(x)$,

(A3) $\forall x \sqsubset \mathsf{a}_3\, h\, z A_{\mathrm{b}}(x) \rightarrow \forall y^\rho \sqsubset z \forall x^\sigma \sqsubset hy A_{\mathrm{b}}(x)$.

(Standard) Proof Translation

Theorem. If

- conditions (A1), (A2), (A3) hold
- $\mathsf{HA}^\omega \vdash A$,

then there is a sequence of terms t such that

$$\mathsf{HA}^\omega \vdash \forall y |A|^t_y.$$

(Standard) Proof Translation

Theorem. If

- conditions (A1), (A2), (A3) hold
- $\mathsf{HA}^\omega + \Delta \vdash A$,

then there is a sequence of terms t such that

$$\mathsf{HA}^\omega \vdash \forall y |A|^t_y,$$

if Δ is such that

$$\mathsf{HA}^\omega \vdash \forall v |\Delta|^q_v$$

for some sequence of terms q

Condition (A2)

- Joining two sets of counter-examples into one

  $\forall x \sqsubset \mathsf{a}_2\, y_0\, y_1\, A_{\mathrm{b}}(x) \to \bigwedge_{i=0}^{1} \forall x \sqsubset y_i A_{\mathrm{b}}(x)$.

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$$\dfrac{\begin{array}{c}[\Gamma]_\alpha \\ \vdots\ \pi_0 \\ A\end{array} \qquad \begin{array}{c}[\Gamma]_\alpha \\ \vdots\ \pi_1 \\ B\end{array}}{A \wedge B}\ \wedge\mathsf{I}$$

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$$\dfrac{\begin{array}{c}[\forall w \sqsubset p\ |\Gamma|_w]_{\alpha_0} \\ \vdots\ \pi_0 \\ |A|^t\end{array} \qquad \begin{array}{c}[\forall w \sqsubset q\ |\Gamma|_w]_{\alpha_1} \\ \vdots\ \pi_1 \\ |B|^s\end{array}}{|A \wedge B|^{t,s}}\ \wedge\mathsf{I}$$

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  $\forall x \sqsubset \mathsf{a}_2\, y_0\, y_1\, A_{\mathrm{b}}(x) \rightarrow \bigwedge_{i=0}^{1} \forall x \sqsubset y_i A_{\mathrm{b}}(x)$.

$$
\dfrac{\dfrac{\dfrac{[\forall w \sqsubset \mathsf{a}_2\, p\, q\, |\Gamma|_w]_\alpha}{\forall w \sqsubset p\, |\Gamma|_w} \quad \vdots\, \pi_0}{|A|^t} \qquad \dfrac{\dfrac{[\forall w \sqsubset \mathsf{a}_2\, p\, q\, |\Gamma|_w]_\alpha}{\forall w \sqsubset q\, |\Gamma|_w}\,(\mathrm{A2}) \quad \vdots\, \pi_1}{|B|^s}}{|A \wedge B|^{t,s}}\ \wedge\mathrm{I}
$$

Characterisation

Let

$$\mathsf{MP}_{\sqsubset} \quad : \quad (\forall x A_{\mathrm{b}}(x) \to B_{\mathrm{b}}) \to \exists b (\forall x \sqsubset b A_{\mathrm{b}}(x) \to B_{\mathrm{b}})$$

$$\mathsf{AC} \quad : \quad \forall x \exists y A(x,y) \to \exists f \forall x A(x, fx)$$

$$\mathsf{IP}_{\sqsubset} \quad : \quad (\forall x A_{\mathrm{b}}(x) \to \exists y B(y)) \to \exists y (\forall x A_{\mathrm{b}}(x) \to B(y))$$

Theorem. $\mathsf{HA}^{\omega} + \mathsf{MP}_{\sqsubset} + \mathsf{AC} + \mathsf{IP}_{\sqsubset} \vdash A \leftrightarrow \exists x \forall y |A|^x_y$

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Parametrised Proof Translation

$$\mathsf{HA}^\omega \vdash A \quad \Rightarrow \quad \mathsf{HA}^\omega \vdash B$$

for some B such that $\mathsf{HA}^\omega \vdash B \to \exists x \forall y |A|^x_y$

Parametrised Proof Translation

$$\text{HA}^\omega \vdash A \quad \Rightarrow \quad \text{HA}^\omega \vdash B$$

for some B such that $\text{HA}^\omega \vdash B \to \exists x \forall y |A|^x_y$

- $B \equiv \forall y |A|^t_y$, for some term t
- $B \equiv \exists x \leq^* t \forall y |A|^t_y$ ($\leq^*$ is Howard/Bezem majorizability)
- $B \equiv \exists x \forall y |A|^x_y$

Parametrised Proof Translation

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- $B \equiv \forall y |A|^t_y$, for some term t
- $B \equiv \exists x \leq^* t \forall y |A|^t_y$ ($\leq^*$ is Howard/Bezem majorizability)
- $B \equiv \exists x \forall y |A|^x_y$

- $B \equiv \exists x \prec t \forall y |A|^x_y$

Condition on Choice of $\exists x \prec a A(x)$

(B) For each formula A, closed term s and term $t[f]$, if

$$\mathsf{HA}^\omega \vdash \exists z \prec s \forall y |A|_y^{t[z]}$$

then there exists a closed term t^* such that

$$\mathsf{HA}^\omega \vdash \exists x \prec t^* \forall y |A|_y^x.$$

Condition on Choice of $\exists x \prec a A(x)$

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$$\mathsf{HA}^\omega \vdash \exists z \prec s \forall y |A|_y^{t[z]}$$

then there exists a closed term t^* such that

$$\mathsf{HA}^\omega \vdash \exists x \prec t^* \forall y |A|_y^x.$$

We call t^* a $\prec$-*majorizing term* for t.

Conditions on $\forall x \sqsubset a A(x)$ and $\exists x \prec a A(x)$

For each $\sqsubset$-bounded formula $A_{\mathrm{b}}(x^\rho)$,

there are terms $\mathsf{a}_1^*, \mathsf{a}_2^*, \mathsf{a}_3^*$ such that

(A1)* $\mathsf{HA}^\omega \vdash \exists \nu \prec \mathsf{a}_1^* \forall y$

$$(\forall x \sqsubset \nu y \, A_{\mathrm{b}}(x) \to A_{\mathrm{b}}(y)),$$

(A2)* $\mathsf{HA}^\omega \vdash \exists \chi \prec \mathsf{a}_2^* \forall y_0, y_1$

$$(\forall x \sqsubset \chi y_0 y_1 \, A_{\mathrm{b}}(x) \to \bigwedge_{i=0}^{1} \forall x \sqsubset y_i A_{\mathrm{b}}(x)),$$

(A3)* $\mathsf{HA}^\omega \vdash \exists \xi \prec \mathsf{a}_3^* \forall h, z$

$$(\forall x^\sigma \sqsubset \xi h z A_{\mathrm{b}}(x) \to \forall y^\tau \sqsubset z \forall x^\sigma \sqsubset h y A_{\mathrm{b}}(x)).$$

Parametrised Proof Translation

Theorem. If

- conditions (A1)*, (A2)*, (A3)* and (B) hold
- HA$^\omega + \Gamma \vdash A$,

then there are sequences of closed terms t, r such that

$$\text{HA}^\omega \vdash \exists f \prec t \exists g \prec r \forall v, y |\Gamma \to A|^{g,f}_{v,y}.$$

Parametrised Proof Translation

Theorem. If

- conditions (A1)*, (A2)*, (A3)* and (B) hold
- $\mathsf{HA}^\omega + \Delta + \Gamma \vdash A$,

then there are sequences of closed terms t, r such that

$$\mathsf{HA}^\omega \vdash \exists f \prec t \exists g \prec r \forall v, y | \Gamma \to A|^{g,f}_{v,y},$$

if Δ is such that

$$\mathsf{HA}^\omega \vdash \exists a \prec q \forall u | \Delta|^{a}_{u},$$

for some closed term q.

Summary

$\forall x \sqsubset a A(x)$	$\exists x \prec a A(x)$	**Interpretation**
$A(a)$	$A(a)$	Dialectica
$\forall x \in a\, A(x)$	$A(a)$	Diller-Nahm
$\forall i^{n-1}\, A(ai) \mid \forall x A(x)$	$A(a)$	Stein's family
$\forall x A(x)$	$A(a)$	Modified realizability
$A(a)$	$\exists x \leq^* a A(x)$	Monotone Dialectica
$\forall x \in a\, A(x)$	$\exists x \leq^* a A(x)$	*no given name*
$\forall i^{n-1}\, A(ai) \mid \forall x A(x)$	$\exists x \leq^* a A(x)$	*no given name*
$\forall x A(x)$	$\exists x \leq^* a A(x)$	Monotone realizability