

# Unifying Functional Interpretations

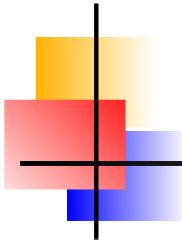
Paulo Oliva

pbo@dcs.qmul.ac.uk

Queen Mary, University of London, UK

*Dagstuhl, January 11, 2004*





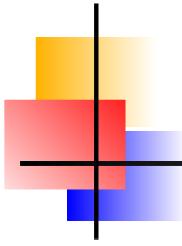
# History

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1958. Gödel's Dialectica interpretation

- *Relative consistency of PA*





# History

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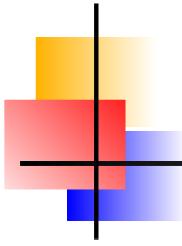
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1959. Kreisel's modified realizability

- *Independence results, unwinding proofs*





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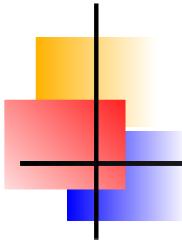
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1974. Diller-Nahm variant of Dialectica interpretation

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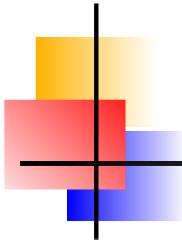
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1978. Stein's family of functional interpretations

- *Relate modified realizability and Diller-Nahm's*





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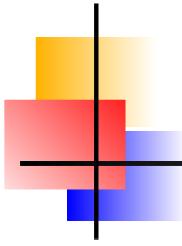
1978. Stein's family of functional interpretations

- *Relate modified realizability and Diller-Nahm's*

1992. Monotone functional interpretation

- *Proof mining*



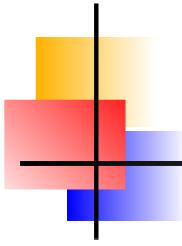


# *Goal*

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*Relation between Dialectica interpretation  
and modified realizability.*





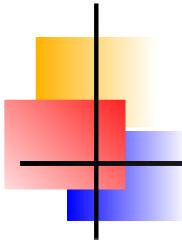
# *Goal*

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*Relation between Dialectica interpretation  
and modified realizability.*

*Common framework for all functional interpretations.*



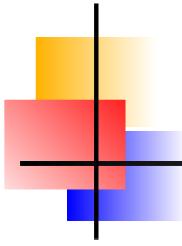


# *Road map*

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1. Functional Interpretations
2. Parametrised Formula Translation
  - (Standard) Proof Translation for PFT
3. Parametrised Proof Translation



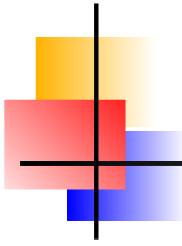


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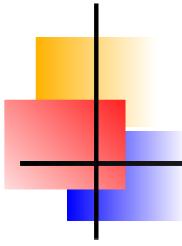


# Logical Framework

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- Language of finite types  $\mathcal{T}$ 
  - $\mathbb{N} \in \mathcal{T}$
  - $\rho, \sigma \in \mathcal{T} \Rightarrow \rho \rightarrow \sigma \in \mathcal{T}$
- Variable and quantifiers for each finite type  $\rho \in \mathcal{T}$ 
  - $\forall x^\rho A(x)$
  - $\exists x^\rho A(x)$
- $\text{HA}^\omega \equiv$  Heyting arithmetic in the language of finite types





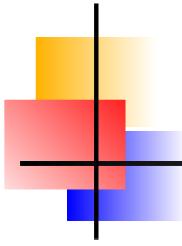
# Logical Framework

- Constructs:

$A \wedge B$	conjunction
$A \vee B$	disjunction
$A \nabla B$	classical disjunction
$\neg A$	negation
$\exists x^\rho A$	existential quantifier
$\forall x^\rho A$	universal quantifier

- $A \rightarrow B \equiv \neg A \vee B$





# A Functional Interpretation

- A formula mapping

$$A \mapsto |A|_y^x$$

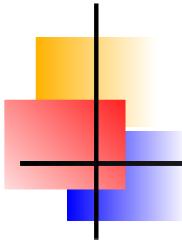
- $x$  marks the **witness** required by  $A$  (i.e.  $\forall y |A|_y^t$ )
- $y$  marks the **refutation** of a given witness for  $A$ .

- A proof mapping

$$\text{HA}^\omega \vdash A \mapsto \text{HA}^\omega \vdash B,$$

for some  $B$  such that  $\text{HA}^\omega \vdash B \rightarrow \exists x \forall y |A|_y^x$ .





# A Functional Interpretation

- A formula mapping

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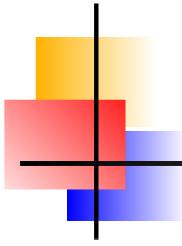
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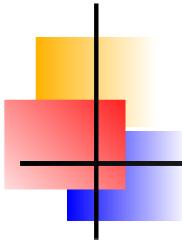


# *Road map*

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1. Functional Interpretations
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  - (Standard) Proof Translation for PFT
3. Parametrised Proof Translation





## The Formula Translation

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$|A_{\text{at}}| := A_{\text{at}}$ , when  $A_{\text{at}}$  is atomic.

Assume we have already defined  $|A|_y^x$  and  $|B|_w^v$ , we define

$$|A \wedge B|_{y,w}^{x,v} := |A|_y^x \wedge |B|_w^v,$$

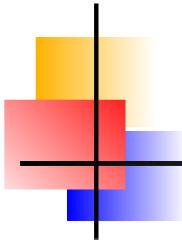
$$|A \vee B|_{y,w}^{x,v,n} := (n = 0 \rightarrow |A|_y^x) \wedge (n \neq 0 \rightarrow |B|_w^v),$$

$$|A \triangleright B|_{y,w}^{f,g} := |A|_y^{gw} \triangleright |B|_w^{fy},$$

$$|\forall z A(z)|_{y,z}^f := |A(z)|_y^{ fz}$$

$$|\exists z A(z)|_y^{x,z} := |A(z)|_y^x.$$





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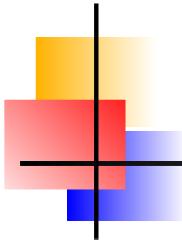
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- What about  $|\neg A|$ ?



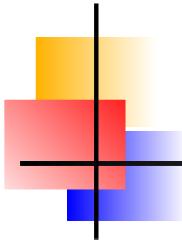


## *The Witnesses of $\neg A$*

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Assume  $A$  has interpretation  $|A|_y^x$





## The Witnesses of $\neg A$

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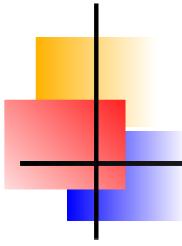
Assume  $A$  has interpretation  $|A|_y^x$

- Gödel's Dialectica interpretation:

Functionals producing counter-examples for  $A$ , i.e.

$$|\neg A|_x^f := \neg |A|_{fx}^x$$





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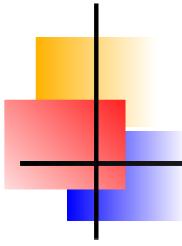
$$|\neg A|_x^f := \neg |A|_{fx}^x$$

- Modified Realizability

$\neg A$  does not ask for witnesses, i.e.

$$|\neg A|_x := \neg \forall y |A|_y^x$$





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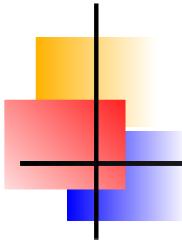
$$|\neg A|_x^f := \neg |A|_{fx}^x \equiv \neg \forall y (\textcolor{red}{y = fx} \rightarrow |A|_y^x)$$

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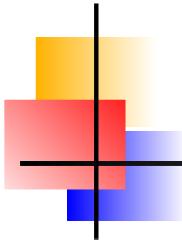
$$|\neg A|_x := \neg\forall y|A|_y^x \equiv \neg\forall y(\textcolor{red}{true} \rightarrow |A|_y^x)$$

- In General:

Functionals producing “bound” on counter-examples

$$|\neg A|_x^f := \neg\forall y \sqsubset fx|A|_y^x \equiv \neg\forall y(\textcolor{red}{y \sqsubset fx} \rightarrow |A|_y^x)$$





# Parametrised Formula Translation

$$|A \wedge B|_{y,w}^{x,v} \quad := \quad |A|_y^x \wedge |B|_w^v$$

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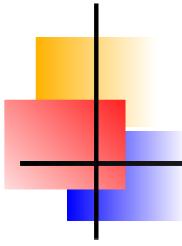
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$$|\forall z A(z)|_{y,z}^f \quad := \quad |A(z)|_y^{fz}$$

$$|\exists z A(z)|_{y,z}^x \quad := \quad |A(z)|_y^x$$

$$|\neg A|_x^f \quad := \quad \neg \forall y \sqsubset fx |A|_y^x$$





# Instantiations

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- **Modified realizability**

Choose  $\forall x \sqsubset aA(x) : \equiv \forall x A(x)$

$$\text{HA}^\omega \vdash \exists x \forall y |A|_y^x \leftrightarrow \exists x (x \text{ mr } A).$$

- **Dialectica interpretation**

Choose  $\forall x \sqsubset aA(x) : \equiv A(a)$

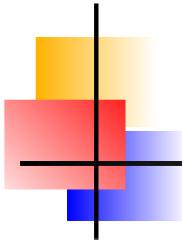
$$\text{HA}^\omega \vdash \exists x \forall y |A|_y^x \leftrightarrow \exists x \forall y A_D(x, y).$$

- **Diller-Nahm variant**

Choose  $\forall x \sqsubset aA(x) : \equiv \forall x \in a A(x)$

$$\text{HA}^\omega \vdash \exists x \forall y |A|_y^x \leftrightarrow \exists x \forall y A_\wedge(x, y).$$





# *The $\sqsubset$ -Bounded Formulas*

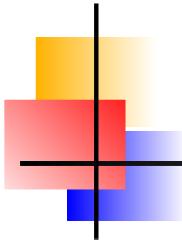
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## **Definition.**

Those built out of prime formulas via

- conjunction ( $A_b \wedge B_b$ )
- implication ( $A_b \rightarrow B_b$ )
- bounded quantification ( $\forall x \sqsubset t A_b$ )





# The $\sqsubset$ -Bounded Formulas

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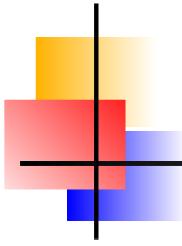
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$\sqsubset$ -bounded formulas  $\equiv \exists$ -free formulas.





# The $\sqsubset$ -Bounded Formulas

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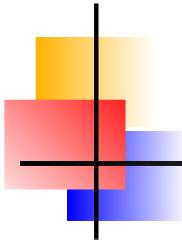
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- conjunction ( $A_b \wedge B_b$ )
- implication ( $A_b \rightarrow B_b$ )
- bounded quantification ( $\forall x \sqsubset t A_b : \equiv A_b[t/x]$ )

$\sqsubset$ -bounded formulas  $\equiv$  quantifier-free formulas.





## *Conditions on Choice of $\forall x \sqsubset a A(x)$*

For each  $\sqsubset$ -bounded formula  $A_b(x^\rho)$ ,

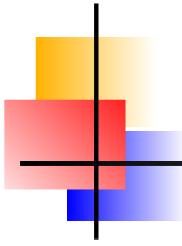
there are terms  $a_1, a_2, a_3$  such that

$$(A1) \quad \forall x \sqsubset a_1 y A_b(x) \rightarrow A_b(y),$$

$$(A2) \quad \forall x \sqsubset a_2 y_0 y_1 A_b(x) \rightarrow \bigwedge_{i=0}^1 \forall x \sqsubset y_i A_b(x),$$

$$(A3) \quad \forall x \sqsubset a_3 h z A_b(x) \rightarrow \forall y^\rho \sqsubset z \forall x^\sigma \sqsubset h y A_b(x).$$





## *(Standard) Proof Translation*

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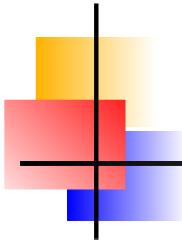
**Theorem.** If

- conditions (A1), (A2), (A3) hold
- $\text{HA}^\omega \vdash A$ ,

then there is a sequence of terms  $t$  such that

$$\text{HA}^\omega \vdash \forall y |A|_y^t.$$





## *(Standard) Proof Translation*

**Theorem.** If

- conditions (A1), (A2), (A3) hold
- $\text{HA}^\omega + \Delta \vdash A$ ,

then there is a sequence of terms  $t$  such that

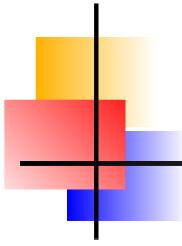
$$\text{HA}^\omega \vdash \forall y |A|_y^t,$$

if  $\Delta$  is such that

$$\text{HA}^\omega \vdash \forall v |\Delta|_v^q$$

for some sequence of terms  $q$





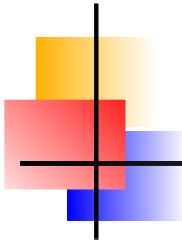
## Condition (A2)

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- Joining two sets of counter-examples into one

$$\forall x \sqsubset a_2 y_0 y_1 A_b(x) \rightarrow \bigwedge_{i=0}^1 \forall x \sqsubset y_i A_b(x).$$





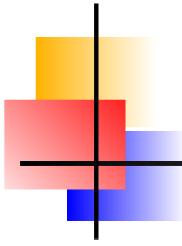
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$$\frac{\begin{array}{c} [\Gamma]_\alpha & [\Gamma]_\alpha \\ \vdots & \vdots \\ \vdots \pi_0 & \vdots \pi_1 \\ A & B \end{array}}{A \wedge B} \wedge I$$



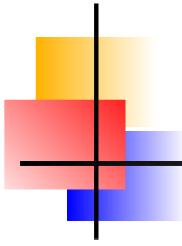


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$$\frac{\begin{array}{c} [\forall w \sqsubset p \mid \Gamma|_w]_{\alpha_0} \quad [\forall w \sqsubset q \mid \Gamma|_w]_{\alpha_1} \\ \vdots \qquad \qquad \qquad \vdots \\ \vdots \pi_0 \qquad \qquad \qquad \vdots \pi_1 \\ |A|^t \qquad \qquad \qquad |B|^s \end{array}}{|A \wedge B|^{t,s}} \wedge I$$

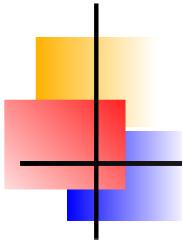


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$$\frac{\frac{[\forall w \sqsubset a_2 p q |\Gamma|_w]_\alpha}{\forall w \sqsubset p |\Gamma|_w} \quad \frac{[\forall w \sqsubset a_2 p q |\Gamma|_w]_\alpha}{\forall w \sqsubset q |\Gamma|_w}}{\begin{array}{c} \vdots \\ \vdots \pi_0 \end{array} \quad \begin{array}{c} \vdots \\ \vdots \pi_1 \end{array}} \text{(A2)} \\ \frac{|A|^t \quad |B|^s}{|A \wedge B|^{t,s}} \wedge |$$



# Characterisation

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Let

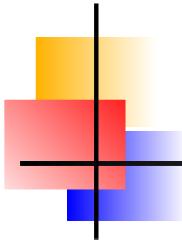
$$\text{MP}_{\sqsubset} : (\forall x A_b(x) \rightarrow B_b) \rightarrow \exists b (\forall x \sqsubset b A_b(x) \rightarrow B_b)$$

$$\text{AC} : \forall x \exists y A(x, y) \rightarrow \exists f \forall x A(x, fx)$$

$$\text{IP}_{\sqsubset} : (\forall x A_b(x) \rightarrow \exists y B(y)) \rightarrow \exists y (\forall x A_b(x) \rightarrow B(y))$$

**Theorem.**  $\text{HA}^\omega + \text{MP}_{\sqsubset} + \text{AC} + \text{IP}_{\sqsubset} \vdash A \leftrightarrow \exists x \forall y |A|_y^x$



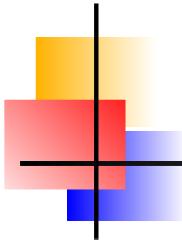


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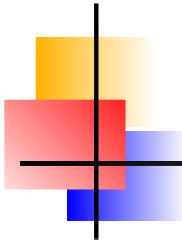
# Parametrised Proof Translation

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$$\text{HA}^\omega \vdash A \quad \Rightarrow \quad \text{HA}^\omega \vdash B$$

for some  $B$  such that  $\text{HA}^\omega \vdash B \rightarrow \exists x \forall y |A|_y^x$





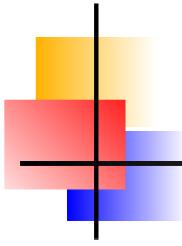
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for some  $B$  such that  $\text{HA}^\omega \vdash B \rightarrow \exists x \forall y |A|_y^x$

- $B \equiv \forall y |A|_y^t$ , for some term  $t$
- $B \equiv \exists x \leq^* t \forall y |A|_y^t$  ( $\leq^*$  is Howard/Bezem majorizability)
- $B \equiv \exists x \forall y |A|_y^x$





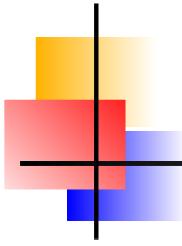
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- $B \equiv \exists x \leq^* t \forall y |A|_y^t$  ( $\leq^*$  is Howard/Bezem majorizability)
- $B \equiv \exists x \forall y |A|_y^x$
  
- $B \equiv \exists x \prec t \forall y |A|_y^x$





## *Condition on Choice of $\exists x \prec aA(x)$*

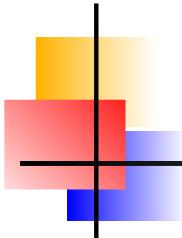
(B) For each formula  $A$ , closed term  $s$  and term  $t[f]$ , if

$$\text{HA}^\omega \vdash \exists z \prec s \forall y |A|_y^{t[z]}$$

then there exists a closed term  $t^*$  such that

$$\text{HA}^\omega \vdash \exists x \prec t^* \forall y |A|_y^x.$$





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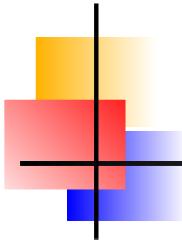
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We call  $t^*$  a  $\prec$ -majorizing term for  $t$ .





## Conditions on $\forall x \sqsubset aA(x)$ and $\exists x \prec aA(x)$

For each  $\sqsubset$ -bounded formula  $A_b(x^\rho)$ ,

there are terms  $a_1^*, a_2^*, a_3^*$  such that

$$(A1)^* \text{ HA}^\omega \vdash \exists \nu \prec a_1^* \forall y$$

$$(\forall x \sqsubset \nu y A_b(x) \rightarrow A_b(y)),$$

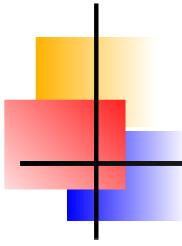
$$(A2)^* \text{ HA}^\omega \vdash \exists \chi \prec a_2^* \forall y_0, y_1$$

$$(\forall x \sqsubset \chi y_0 y_1 A_b(x) \rightarrow \bigwedge_{i=0}^1 \forall x \sqsubset y_i A_b(x)),$$

$$(A3)^* \text{ HA}^\omega \vdash \exists \xi \prec a_3^* \forall h, z$$

$$(\forall x^\sigma \sqsubset \xi h z A_b(x) \rightarrow \forall y^\tau \sqsubset z \forall x^\sigma \sqsubset h y A_b(x)).$$





# Parametrised Proof Translation

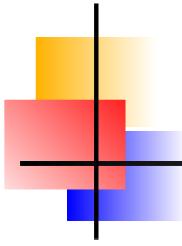
**Theorem.** If

- conditions (A1)\*, (A2)\*, (A3)\* and (B) hold
- $\text{HA}^\omega + \Gamma \vdash A$ ,

then there are sequences of closed terms  $t, r$  such that

$$\text{HA}^\omega \vdash \exists f \prec t \exists g \prec r \forall v, y | \Gamma \rightarrow A|_{v,y}^{g,f}.$$





# Parametrised Proof Translation

**Theorem.** If

- conditions (A1)\*, (A2)\*, (A3)\* and (B) hold
- $\text{HA}^\omega + \Delta + \Gamma \vdash A$ ,

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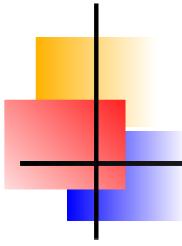
$$\text{HA}^\omega \vdash \exists f \prec t \exists g \prec r \forall v, y | \Gamma \rightarrow A|_{v,y}^{g,f},$$

if  $\Delta$  is such that

$$\text{HA}^\omega \vdash \exists a \prec q \forall u | \Delta|_u^a,$$

for some closed term  $q$ .





# Summary

$\forall x \sqsubset aA(x)$	$\exists x \prec aA(x)$	<b>Interpretation</b>
$A(a)$	$A(a)$	Dialectica
$\forall x \in a A(x)$	$A(a)$	Diller-Nahm
$\forall i^{n-1} A(ai) \mid \forall x A(x)$	$A(a)$	Stein's family
$\forall x A(x)$	$A(a)$	Modified realizability
$A(a)$	$\exists x \leq^* aA(x)$	Monotone Dialectica
$\forall x \in a A(x)$	$\exists x \leq^* aA(x)$	<i>no given name</i>
$\forall i^{n-1} A(ai) \mid \forall x A(x)$	$\exists x \leq^* aA(x)$	<i>no given name</i>
$\forall x A(x)$	$\exists x \leq^* aA(x)$	Monotone realizability

