Unifying Functional Interpretations

Paulo Oliva

pbo@dcs.qmul.ac.uk

Queen Mary, University of London, UK

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Road map

- Functional Interpretations
- Formula Translation Parametrised
- Soundness Theorem
- Proof Translation Parametrised
- Parametrised Soundness Theorem

Road map

- Functional Interpretations
  - Brief History
  - Formal System
  - Formula/Proof Translations
- Formula Translation Parametrised
- Soundness Theorem
- Proof Translation Parametrised
- Parametrised Soundness Theorem

History

1958. Gödel's Dialectica interpretation

- *Relative consistency of* PA

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1992. Monotone functional interpretation

- *Proof mining*

Basic Logical Symbols

- Our logical constructions are:

$A \wedge B$	conjunction
$A \triangledown B$	classical disjunction
$\neg A$	negation
$\exists x A$	existential quantifier
$\forall x A$	universal quantifier

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- $A \to B \equiv \neg A \triangledown B$
- $A \vee B \equiv \exists n((n = 0 \to A) \wedge (n \neq 0 \to B))$.

A Functional Interpretation

- A formula mapping

$$A \quad \mapsto \quad |A|^x_y$$

  - x marks the witness required by A
  - y marks the refutation of a given witness for A.

- A proof mapping

$$\mathsf{HA}^\omega \vdash A \quad \mapsto \quad \mathsf{HA}^\omega \vdash B,$$

for some B such that $\mathsf{HA}^\omega \vdash B \to \exists x \forall y |A|^x_y$.

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- Functional Interpretations
- Formula Translation Parametrised
  - Parametrisation
  - Conditions on Parameter
- Soundness Theorem
- Proof Translation Parametrised
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The Formula Translation

$|A_{\mathsf{at}}| :\equiv A_{\mathsf{at}}$, when A_{at} is atomic.

Assume we have already defined $|A|^x_y$ and $|B|^v_w$, we define

$$
\begin{aligned}
|A \wedge B|^{x,v}_{y,w} &:\equiv |A|^x_y \wedge |B|^v_w, \\
|A \triangledown B|^{f,g}_{y,w} &:\equiv |A|^{gw}_y \triangledown |B|^{fy}_w, \\
|\forall z A(z)|^f_{y,z} &:\equiv |A(z)|^{fz}_y \\
|\exists z A(z)|^{x,z}_y &:\equiv |A(z)|^x_y.
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- What about $|\neg A|$?

The Witnesses of $\neg A$

- Gödel's Dialectica interpretation:

  Functionals producing counter-examples for A, i.e.

  $|\neg A|^f_x :\equiv \neg|A|^x_{fx}$.

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- Modified Realizability

  $\neg A$ does not ask for witnesses, it is either true or false, i.e.

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- In General:

  Functionals producing "bound" on counter-examples

  $|\neg A|^f_x :\equiv \neg \forall y \sqsubset fx |A|^x_y \equiv \neg \forall y (y \sqsubset fx \to |A|^x_y)$.

Formula Translation: Parametrisation

$$
\begin{aligned}
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|\neg A|^{f}_{x} & :\equiv & \neg \forall y \sqsubset fx |A|^x_y.
\end{array}
$$

- For instance:

$$
\begin{array}{rcr}
|x = 0| & \equiv & (x = 0), \\
|\forall x (x = 0)|_a & \equiv & (a = 0), \\
|\neg \forall x (x = 0)|^b & \equiv & \neg \forall a \sqsubset b (a = 0).
\end{array}
$$

Instantiations

- **Modified realizability**

  Choose $\forall x \sqsubset a A(x) :\equiv \forall x A(x)$

  $\mathsf{HA}^\omega \vdash x \text{ mr } A \leftrightarrow \forall y |A|^x_y.$

- **Dialectica interpretation**

  Choose $\forall x \sqsubset a A(x) :\equiv A(a)$

  $\mathsf{HA}^\omega \vdash A_D(x, y) \leftrightarrow |A|^x_y.$

- **Diller-Nahm variant**

  Choose $\forall x \sqsubset a A(x) :\equiv \forall n \leq |a| A(a_n)$

  $\mathsf{HA}^\omega \vdash A_\wedge(x, y) \leftrightarrow |A|^x_y.$

The $\sqsubset$-Bounded Formulas

Definition. The class of $\sqsubset$-*bounded formulas* (we use $A_{\rm b}$ and $B_{\rm b}$) are those built out of

- prime formulas,
- conjunction $(A_{\rm b} \wedge B_{\rm b})$,
- implication $(A_{\rm b} \to B_{\rm b})$ and
- bounded quantification $(\forall x \sqsubset t A_{\rm b})$.

Conditions on Choice of $\forall x \sqsubset a A(x)$

For each $\sqsubset$-bounded formula $A_{\mathrm{b}}(x^\rho)$, there are terms b_1 , b_2 , b_3 such that

(B1) $\mathsf{HA}^\omega \vdash \forall x \sqsubset \mathsf{b}_1\, y\, A_{\mathrm{b}}(x) \to A_{\mathrm{b}}(y)$,

(B2) $\mathsf{HA}^\omega \vdash \forall x \sqsubset \mathsf{b}_2\, y_0\, y_1\, A_{\mathrm{b}}(x) \to \bigwedge_{i=0}^{1} \forall x \sqsubset y_i A_{\mathrm{b}}(x)$,

(B3) $\mathsf{HA}^\omega \vdash \forall x \sqsubset \mathsf{b}_3\, h\, z A_{\mathrm{b}}(x) \to \forall y^\rho \sqsubset z \forall x^\sigma \sqsubset h y A_{\mathrm{b}}(x)$.

Soundness Theorem

Soundness Theorem. If

- conditions (B1), (B2), (B3) hold
- $\mathsf{HA}^\omega + \Gamma \vdash A$,

then there are sequences of terms $t[v]$ and $r[v, y]$ such that

$$\mathsf{HA}^\omega + \forall w \sqsubset r[v, y] |\Gamma|^v_w \vdash |A|^{t[v]}_y.$$

Condition (B1)

- Any counter-example has a bound

  $\forall y' \sqsubset \mathsf{b}_1\, y\, A_{\mathrm{b}}(y') \to A_{\mathrm{b}}(y)$.

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$$[A]_\alpha$$

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$$\frac{[\forall y' \sqsubset \mathsf{b}_1\, y\, |A|^x_y]_\alpha}{|A|^x_y}\ \text{(B1)}$$

Condition (B2)

- Joining two sets of counter-examples into one

  $\forall x \sqsubset \mathsf{b}_2\, y_0\, y_1\, A_{\mathrm{b}}(x) \rightarrow \bigwedge_{i=0}^{1} \forall x \sqsubset y_i A_{\mathrm{b}}(x)$.

Condition (B2)

- Joining two sets of counter-examples into one

  $\forall x \sqsubset \mathsf{b}_2\, y_0\, y_1\, A_{\mathrm{b}}(x) \to \bigwedge_{i=0}^{1} \forall x \sqsubset y_i A_{\mathrm{b}}(x)$.

$$\dfrac{\begin{array}{c}[\Gamma]_\alpha \\ \vdots\ \pi_0 \\ A\end{array} \qquad \begin{array}{c}[\Gamma]_\alpha \\ \vdots\ \pi_1 \\ B\end{array}}{A \wedge B}\ \wedge\mathsf{I}$$

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$$\dfrac{\begin{array}{c}[\forall w \sqsubset p\ |\Gamma|_w]_{\alpha_0} \\ \vdots\ \pi_0 \\ |A|^t\end{array} \qquad \begin{array}{c}[\forall w \sqsubset q\ |\Gamma|_w]_{\alpha_1} \\ \vdots\ \pi_1 \\ |B|^s\end{array}}{|A \wedge B|^{t,s}}\ \wedge\mathsf{I}$$

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Condition (B3)

- Bounded family of sets into a single set

$$\forall x \sqsubset \mathsf{b}_3\, h\, z A_{\rm b}(x) \to \forall y^\rho \sqsubset z \forall x^\sigma \sqsubset h y A_{\rm b}(x).$$

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$$\dfrac{\begin{array}{c}[\Gamma]_\alpha \\ \vdots\ \pi \\ A\end{array} \quad A \to B}{B} \to \mathsf{E}$$

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$$\dfrac{\begin{array}{c}\forall u \sqsubset ry|\Gamma|_u \\ \vdots\ \tilde{\pi} \\ |A|^s_y\end{array} \qquad \forall y \sqsubset q|A|^x_y \to |B|^t_w}{|B|^{ts}_w} \to \mathsf{E}$$

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$$\dfrac{\dfrac{[\forall y \sqsubset q(\forall u \sqsubset ry|\Gamma|_u)]_\alpha}{\vdots\; \tilde{\pi}} \\ \forall y \sqsubset q(\,|A|_y^s\,) \qquad \forall y \sqsubset q|A|_y^x \to |B|_w^t}{|B|_w^{ts}} \to \mathsf{E}$$

Condition (B3)

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$$\dfrac{\dfrac{\dfrac{[\forall u \sqsubset \mathsf{b}_3\, r\, q |\Gamma|_u]_\alpha}{\forall y \sqsubset q(\forall u \sqsubset ry|\Gamma|_u)}\,(\mathrm{B3})}{\forall y \sqsubset q(|A|^s_y)}\tilde{\pi} \qquad \forall y \sqsubset q|A|^x_y \to |B|^t_w}{|B|^{ts}_w} \to \mathrm{E}$$

Necessity of (B) conditions

Lemma. If $\mathsf{HA}^\omega + \Gamma \vdash A$ implies the existence of terms $t[v]$ and $r[v,y]$ satisfying

$$\mathsf{HA}^\omega + \forall w \sqsubset r[v,y] |\Gamma|^v_w \vdash |A|^{t[v]}_y,$$

then conditions (B1), (B2) and (B3) must hold for the choice of $\forall x \sqsubset a A(x)$.

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Proof Translation : Parametrisation

If $\mathsf{HA}^\omega \vdash A$ then

$\mathsf{HA}^\omega \vdash B$	and	$\mathsf{HA}^\omega \vdash B \to \exists x \forall y |A|^x_y$
$\mathsf{HA}^\omega \vdash \forall y |A|^t_y$	and	$\mathsf{HA}^\omega \vdash \forall y |A|^t_y \to \exists x \forall y |A|^x_y$
$\mathsf{HA}^\omega \vdash \exists x \forall y |A|^x_y$	and	$\mathsf{HA}^\omega \vdash \exists x \forall y |A|^x_y \to \exists x \forall y |A|^x_y$
$\mathsf{HA}^\omega \vdash \exists x \leq^* t \forall y |A|^x_y$	and	$\mathsf{HA}^\omega \vdash \exists x \leq^* t \forall y |A|^x_y \to \exists x \forall y |A|^x_y$

Proof Translation : Parametrisation

If HA$^\omega \vdash A$ then

HA$^\omega \vdash B$	and	HA$^\omega \vdash B \to \exists x \forall y |A|^x_y$
HA$^\omega \vdash \forall y |A|^t_y$	and	HA$^\omega \vdash \forall y |A|^t_y \to \exists x \forall y |A|^x_y$
HA$^\omega \vdash \exists x \forall y |A|^x_y$	and	HA$^\omega \vdash \exists x \forall y |A|^x_y \to \exists x \forall y |A|^x_y$
HA$^\omega \vdash \exists x \leq^* t \forall y |A|^x_y$	and	HA$^\omega \vdash \exists x \leq^* t \forall y |A|^x_y \to \exists x \forall y |A|^x_y$
HA$^\omega \vdash \exists x \prec t \forall y |A|^x_y$	and	HA$^\omega \vdash \exists x \prec t \forall y |A|^x_y \to \exists x \forall y |A|^x_y$

Condition on Choice of $\exists x \prec a A(x)$

For each $\sqsubset$-bounded formula $A_{\rm b}(a, x)$ and term $t[v]$ there exists a term $t^*[v]$ such that,

(E) $\mathsf{HA}^\omega \vdash \exists v' \prec v \forall a A_{\rm b}(a, t[v'])) \to \exists x \prec t^*[v] \forall a A_{\rm b}(a, x)$.

Condition on Choice of $\exists x \prec a A(x)$

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We call $t^*[v]$ a $\prec$-*majorizing term* for $t[v]$.

In particular, when t is a closed term we have

$$\mathsf{HA}^\omega \vdash \forall a A_{\mathrm{b}}(a, t) \rightarrow \exists x \prec t^* \forall a A_{\mathrm{b}}(a, x).$$

Conditions on $\forall x \sqsubset a A(x)$ and $\exists x \prec a A(x)$

For each $\sqsubset$-bounded formula $A_{\mathrm{b}}(x^\rho)$, there are terms $\mathsf{b}_1^*, \mathsf{b}_2^*, \mathsf{b}_3^*$ such that

(B1)* $\mathsf{HA}^\omega \vdash \exists \nu \prec \mathsf{b}_1^* \forall a, y$

$$(\forall x \sqsubset \nu a y \, A_{\mathrm{b}}(a, x) \to A_{\mathrm{b}}(a, y)),$$

(B2)* $\mathsf{HA}^\omega \vdash \exists \chi \prec \mathsf{b}_2^* \forall a, y_0, y_1$

$$(\forall x \sqsubset \chi a y_0 y_1 \, A_{\mathrm{b}}(a, x) \to \bigwedge_{i=0}^{1} \forall x \sqsubset y_i A_{\mathrm{b}}(a, x)),$$

(B3)* $\mathsf{HA}^\omega \vdash \exists \xi \prec \mathsf{b}_3^* \forall a, h, z$

$$(\forall x^\sigma \sqsubset \xi a h z A_{\mathrm{b}}(a, x) \to \forall y^\tau \sqsubset z \forall x^\sigma \sqsubset h y A_{\mathrm{b}}(a, x)).$$

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Parametrised Soundness

Parametrised Soundness. If

- conditions (E), (B1)*, (B2)* and (B3)* hold
- HA$^\omega + \Gamma \vdash A$,

then there are sequences of closed terms t, r such that

$$\mathrm{HA}^\omega \vdash \exists f \prec t \exists g \prec r \forall a, v, y(\forall w \sqsubset gavy |\Gamma|^v_w \to |A|^{fav}_y).$$

Summary

$\forall x^\rho \sqsubset a A(x)$	$\exists x \prec a A(x)$	**Interpretation**
$A(a)$	$A(a)$	Dialectica
$\forall i \leq \lvert a \rvert \, A(a_i)$	$A(a)$	Diller-Nahm
$\forall i^{n-1} \, A(ai) \backslash \forall x A(x)$	$A(a)$	Stein's family
$\forall x A(x)$	$A(a)$	Modified realizability
$A(a)$	$\exists x \leq^* a A(x)$	Monotone Dialectica
$\forall i \leq \lvert a \rvert \, A(a_i)$	$\exists x \leq^* a A(x)$	*no given name*
$\forall i^{n-1} \, A(ai) \backslash \forall x A(x)$	$\exists x \leq^* a A(x)$	*no given name*
$\forall x A(x)$	$\exists x \leq^* a A(x)$	Monotone realizability