On the Different Interpretations of Classical Countable Choice

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Road Map

- Introduction
  - Proof Interpretations
  - Classical Logic
  - Countable Choice
- Interpreting Classical Countable Choice
  - Bar Recursion (Spector'62)
  - Modified Bar Recursion (BO'03)
  - Krivine's Clock (Krivine'03)
- Final Remarks

Proof Interpretations

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  - Hypothesis elimination ($\mathcal{T} \equiv \mathcal{S} + P$, $A^I \equiv A$)

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- Associates formula A with $A^I \equiv \exists x A_I(x)$.
- A is interpretable in $\mathcal{T}$ if

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- $|A| \equiv \{t : \mathcal{T} \vdash A_I(t)\}$. (realisers of A)

Proof Interpretations

- Finite type approach ($|A|$ is a set of terms)

$$
\begin{aligned}
|A \rightarrow B| &\equiv |A| \rightarrow |B| \\
|\forall x^\tau A_x| &\equiv \Pi_{x \in \tau} |A_x| \\
|\exists x^\tau A_x| &\equiv \Sigma_{x \in \tau} |A_x|
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- E.g. AC $\equiv \forall x^\sigma \exists y^\tau A(x,y) \to \exists f \forall x A(x, f(x))$

$$|\text{AC}| \equiv \Pi_x \Sigma_y |A(x,y)| \to \Sigma_f \Pi_x |A(x, f(x))|$$

then $\lambda\phi.\langle \lambda x.\pi_0(\phi x), \lambda x.\pi_1(\phi x)\rangle \in |\text{AC}|$.

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- Second order approach (parametrised by model $\mathcal{M}$)

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|\forall x A| &\equiv \bigcap_{a \in \mathcal{M}} |A_a| \\
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\end{aligned}
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Proof Interpretations

- E.g. Let

$$\mathsf{Int}(n) \equiv \forall X(X(0) \wedge \forall x(X(x) \to X(x+1)) \to X(n))$$

then $\lambda x \lambda f.f^n(x) \in |\mathsf{Int}(n)|$.

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Dealing with Countable Choice

cAC : $\forall n \exists y^{\tau} A(n, y) \rightarrow \exists f \forall n A(n, fn)$

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- Accepted intuitionistically.
- Heyting arithmetic closed under "rule of choice":

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  LEM + cAC $\vdash$ CA

  where CA $\equiv \exists f \forall n (f(n) = 0 \leftrightarrow A(n))$.

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- Very strong in the presence of classical logic:

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  where CA $\equiv \exists f \forall n (f(n) = 0 \leftrightarrow A(n))$.
- So, PA^{ω} + cAC $\equiv$ full classical analysis.

Dealing with Classical Logic

- Eliminate classical logic:

  $\mathsf{HA}^\omega + \mathsf{LEM} \vdash A \quad \Rightarrow \quad \mathsf{HA}^\omega \vdash A^N.$

  $N \equiv$ negative translation. (Gödel'33)

- Interpret classical logic:

  Find functional C witnessing LEM^I.

  $C \equiv$ continuation. (Griffin'90)

Interpreting Countable Choice (1)

- Finite type approach

$$\mathsf{HA}^\omega + \mathsf{LEM} + \mathsf{cAC} \vdash A$$

$$\mathsf{HA}^\omega + \mathsf{cAC}^N \vdash A^N \qquad \text{(by negative translation)}$$

$$\mathsf{HA}^\omega + \mathsf{DNS} + \mathsf{cAC} \vdash A^N \qquad \text{(since } \mathsf{DNS} + \mathsf{cAC} \vdash \mathsf{cAC}^N\text{)}$$

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- Bar recursion interprets DNS (Spector'62).

(via functional interpretation (Gödel'56))

Interpreting Countable Choice (2)

Thm(BO'02) Modified bar recursion interprets DNS

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Thm(BO'02) SBR is primitive recursively definable in MBR, but not conversely.

Krivine's Interpretation

- Reduce

$$\forall n \exists S A(n,S) \to \exists Y_{(\cdot)} \forall n A(n, Y_n)$$

to

$$\forall n \exists S A(n,S) \to \exists Y_{(\cdot,\cdot)} \forall n \exists k A(n, Y_{n,k})$$

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which follows from

$$\exists Y_{(\cdot,\cdot)} \forall n (\forall k A(n, Y_{n,k}) \to \forall S A(n, S)).$$

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asks for term χ in the set

$$\bigcup_{Y_{(\cdot,\cdot)}} \bigcap_{n} (\bigcap_{k} (|\mathsf{Int}(k)| \to |A(n, Y_{n,k})|) \to |\forall S A(n, S)|).$$

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- Assume $b : k \mapsto t_k$. Take $Y_{n,k}$ (by cAC) satisfying:

$(*)$ if $t_k(k) \in |A(n, Y_{n,k})|$ then $t_k(k) \in |\forall S A(n, S)|$.

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- We can take $\chi t = t(k_t)$. $\quad$ ($b^{-1} : t \mapsto k_t$)

Comparing the Two Approaches

	Finite types	Second order
Basic terms	Primitive rec	Recursive
Induction	Interprets	Eliminates
Scales	Yes	?
Curry-Howard	?	Yes
Classical Logic	Eliminates	Interprets
Comprehension	Goal	Axiom
Reduces cAC to	DNS	KA

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- Polymorphism versus Bar recursion?