Bounded Functional Interpretation

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BRICS - University of Aarhus

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Context

- Proof mining:

  Logical analysis of (ineffective) mathematical proofs with the aim of extracting new information.

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Context

- Proof mining:

  Logical analysis of (ineffective) mathematical proofs with the aim of extracting new information.

- New information:

  (bound on) witness for existential quantifier.

Driving Idea

If one is looking for bounds, then bounded quantifiers shouldn't have computational content.

Bounded Quantifiers

- $\forall x^{\mathbb{N}} \leq t A(x)$ intrinsically different from $\forall x^{\mathbb{N}} A(x)$.
  - Induction on NP-predicates.
  - Bounded arithmetic.

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Bounded Quantifiers

- $\forall x^{\mathbb{N}} \leq t A(x)$ intrinsically different from $\forall x^{\mathbb{N}} A(x)$.
  - Induction on NP-predicates.
  - Bounded arithmetic.

- $\forall x^{\mathbb{N} \to \mathbb{N}} \leq t A(x)$ intrinsically different from $\forall x^{\mathbb{N} \to \mathbb{N}} A(x)$.
  - $\forall x^{\mathbb{N} \to \mathbb{N}} \leq t \ldots$ $\qquad$ $\forall x \in [0, 1] \ldots$
  - $\forall x^{\mathbb{N} \to \mathbb{N}} \ldots$ $\qquad$ $\forall x \in \mathbb{R} \ldots$

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Bounded Quantifiers

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  - Induction on NP-predicates.
  - Bounded arithmetic.

- $\forall x^{\mathbb{N}\to\mathbb{N}} \leq tA(x)$ intrinsically different from $\forall x^{\mathbb{N}\to\mathbb{N}} A(x)$.
  - $\forall x^{\mathbb{N}\to\mathbb{N}} \leq t \ldots$ $\forall x \in$ compact Polish space.
  - $\forall x^{\mathbb{N}\to\mathbb{N}} \ldots$ $\forall x \in$ Polish space.

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An Example

- RCA_0: Basic theory of analysis.
- WKL: Every infinite binary tree has an infinite path.

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- **Thm**(Kohlenbach'92)

  If $\mathsf{RCA}_0 + \mathsf{WKL} \vdash \forall x \in P; y \in K_x \exists z^\tau A_\exists(x, y, z)$ then

  $\exists$ closed term s s.t. $\forall x \in P; y \in K_x \exists z \leq s(x) A_\exists(x, y, z)$.

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  If $\mathrm{RCA}_0 + \mathrm{WKL} \vdash \forall x^{\mathbb{N}\to\mathbb{N}}; y \leq t(x) \exists z^\tau A_\exists(x, y, z)$ then

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Goal

- Interpretation distinguishing $\forall x^\rho \leq t A(x)$ and $\forall x^\rho A(x)$.

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- Interpretation distinguishing $\forall x^\rho \leq t A(x)$ and $\forall x^\rho A(x)$.
- Need: bounded quantifiers for all finite types.
- Need: extended $\leq$ to all finite types.

One Solution: Pointwise

- Use pointwise less-than-equal-to relation:

$$x \leq_{\rho \to \sigma} y :\equiv \forall z^{\rho} (x(z) \leq_{\sigma} y(z)).$$

One Solution: Pointwise

- Use pointwise less-than-equal-to relation:

$$x \leq_{\rho \to \sigma} y := \forall z^{\rho}(x(z) \leq_{\sigma} y(z)).$$

- Problem:

  $x' \leq x$ and $y' \leq y$ does not imply $x'(y') \leq x(y)$.

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Another Solution: Monotone

- Howard/Bezem's strong majorizability relation.
- Extension of the $\leq$-relation to higher types:
  - $x \leq^*_{\mathbb{N}} y :\equiv x \leq_{\mathbb{N}} y$
  - $x \leq^*_{\rho \to \sigma} y :\equiv \forall v^\rho \forall u \leq^*_\rho v (\underbrace{xu \leq^*_\sigma yv}_{\text{above}} \wedge \underbrace{yu \leq^*_\sigma yv}_{\text{monotone}})$

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  - $x \leq^*_{\rho \to \sigma} y := \forall v^\rho \forall u \leq^*_\rho v (\underbrace{xu \leq^*_\sigma yv}_{\text{above}} \wedge \underbrace{yu \leq^*_\sigma yv}_{\text{monotone}})$
- Example 1: for type $\mathbb{N} \to \mathbb{N}$ we have:

$$f \leq^*_{\mathbb{N} \to \mathbb{N}} g := \forall m \forall n \leq m (\underbrace{f(n) \leq g(m)}_{\text{above}} \wedge \underbrace{g(n) \leq g(m)}_{\text{monotone}}).$$

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- Example 2: $x \leq^*_{\rho\to\sigma} x$ means that x is monotone

$x \leq^*_{\rho\to\sigma} x := \forall v^\rho \forall u \leq^*_\rho v(xu \leq^*_\sigma xv)$.

Majorizability: Some Properties

- $x \leq^* y \wedge y \leq^* z \to x \leq^* z$
- $x' \leq^* x \wedge y' \leq^* y \to x'(y') \leq^* x(y)$
- $x \leq^* y \to y \leq^* y$

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- $x \leq^* y \wedge y \leq^* z \rightarrow x \leq^* z$
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Moreover, for each closed term t of e.g. HA^ω there is another closed term t^* such that, $\mathsf{HA}^\omega \vdash t \leq^* t^*$.

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Majorizability: A New Symbol

- Idea: Add majorizability relation $\trianglelefteq$ to the language, functional interpretation can access the relation.
- Cannot just take:
  - $x \trianglelefteq_{\mathbb{N}} y \leftrightarrow x \leq_{\mathbb{N}} y$
  - $x \trianglelefteq_{\rho\to\sigma} y \leftrightarrow \forall v^\rho \forall u^\rho \trianglelefteq_\rho v (xu \trianglelefteq_\sigma yv \wedge yu \trianglelefteq_\sigma yv)$

  functional interpretation would ask for a witness for

$$x \trianglelefteq_{\rho\to\sigma} y \leftarrow \forall v^\rho \forall u^\rho \trianglelefteq_\rho v (xu \trianglelefteq_\sigma yv \wedge yu \trianglelefteq_\sigma yv)$$

Majorizability: A New Symbol

- Idea: Add majorizability relation $\trianglelefteq$ to the language, functional interpretation can access the relation.
- One solution, use a rule instead of the implication:
  - $x \trianglelefteq_{\mathbb{N}} y \leftrightarrow x \leq_{\mathbb{N}} y$
  - $x \trianglelefteq_{\rho\to\sigma} y \to \forall v^{\rho} \forall u^{\rho} \trianglelefteq_{\rho} v(xu \trianglelefteq_{\sigma} yv \wedge yu \trianglelefteq_{\sigma} yv)$

$$\frac{A_{\mathrm{b}} \to \forall u \trianglelefteq v(su \trianglelefteq tv \wedge tu \trianglelefteq tv)}{A_{\mathrm{b}} \to s \trianglelefteq t}$$

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The Basic Setting

- With the intensional symbol $\trianglelefteq$ we are in the position to define e.g. a "bounded quantifier" of arbitrary type:

$$\mathrm{B}_\forall \;:\; \forall x \trianglelefteq t A(x) \leftrightarrow \forall x (x \trianglelefteq t \to A(x))$$

$$\mathrm{B}_\exists \;:\; \exists x \trianglelefteq t A(x) \leftrightarrow \exists x (x \trianglelefteq t \wedge A(x)).$$

- Let the theory $\mathrm{IL}^\omega_\trianglelefteq$ be intuitionistic logic (in all finite types) plus the axioms/rule for $\trianglelefteq$, $\mathrm{B}_\forall$ and $\mathrm{B}_\exists$.

Monotone Quantifiers

- Quantify over “monotone functionals” as

$$\forall x(x \trianglelefteq x \to A(x))$$

$$\exists x(x \trianglelefteq x \wedge A(x))$$

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Monotone Quantifiers

- Quantify over "monotone functionals" as

  $\forall x(x \trianglelefteq x \to A(x))$

  $\exists x(x \trianglelefteq x \wedge A(x))$

- Use the following abbreviations:

  $\tilde{\forall} x A(x)$ instead of $\forall x(x \trianglelefteq x \to A(x))$

  $\tilde{\exists} x A(x)$ instead of $\exists x(x \trianglelefteq x \wedge A(x))$

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The Interpretation

- Main idea:

$$\text{View } \forall x A(x) \text{ as } \quad \tilde{\forall} b \quad \forall x \trianglelefteq b A(x).$$

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$$\text{View } \forall x A(x) \text{ as } \underbrace{\tilde{\forall} b}_{\text{bounding}} \overbrace{\forall x \trianglelefteq b}^{\text{bounded}} A(x).$$

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- A relativization to Bezem's model $\mathcal{M}$:

$\tilde{\forall} b \forall x \trianglelefteq b A(x)$

$\tilde{\forall} b \forall x (x \trianglelefteq b \to A(x))$

$\forall x (\tilde{\exists} b (x \trianglelefteq b) \to A(x))$

$\forall x (x \in \mathcal{M} \to A(x))$

The Interpretation

- Associate arbitrary formulas of $\mathcal{L}^{\omega}_{\unlhd}$ to formulas having the form $\tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b,c)$.

$$A \in \mathcal{L}^{\omega}_{\unlhd} \qquad \mapsto \qquad (A)^{\mathrm{B}} \equiv \tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b,c).$$

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- Compare with Gödel's functional interpretation

$$A \in \mathcal{L}^{\omega} \qquad \mapsto \qquad (A)^{\mathrm{D}} \equiv \exists x \forall y A_{\mathsf{qf}}(x, y).$$

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- Resulting matrix monotone on the first argument, i.e.

$$b \trianglelefteq b' \wedge c \trianglelefteq c \wedge A_{\mathrm{B}}(b, c, x) \rightarrow A_{\mathrm{B}}(b', c, x)$$

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The Interpretation: Bounded quantifiers

- Assume $(A(x))^{\mathrm{B}} \equiv \tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b, c, x)$.
- $(\forall x \trianglelefteq t A(x))^{\mathrm{B}} \equiv \tilde{\exists} b \tilde{\forall} c \forall x \trianglelefteq t A_{\mathrm{B}}(b, c, x)$.

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(\forall x \trianglelefteq t A(x))^{\mathrm{B}} &\equiv \forall x \trianglelefteq t \tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b, c, x) \\
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The Interpretation: Implication

- Assume $(A)^{\mathrm{B}} \equiv \tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b, c)$ and $(B)^{\mathrm{B}} \equiv \tilde{\exists} d \tilde{\forall} e B_{\mathrm{B}}(d, e)$.
- $(A \to B)^{\mathrm{B}} \equiv \tilde{\exists} f, g \tilde{\forall} b, e(\tilde{\forall} c \trianglelefteq gbe A_{\mathrm{B}}(b, c) \to B_{\mathrm{B}}(fb, e))$

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\end{array}
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The Soundness Theorem

- Let $A(\underline{z})$ be an arbitrary formula of $\mathcal{L}^{\omega}_{\trianglelefteq}$ and assume $(A(\underline{z}))^{\mathrm{B}} \equiv \tilde{\exists} b \tilde{\forall} c A_{\mathrm{B}}(b, c, \underline{z})$.

Thm (Soundness I) If

$$\mathrm{IL}^{\omega}_{\trianglelefteq} \vdash A(\underline{z}),$$

then there are closed monotone terms $\underline{t}$ of appropriate types such that

$$\mathrm{IL}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall} \underline{a} \forall \underline{z} \trianglelefteq \underline{a} \tilde{\forall} c A_{\mathrm{B}}(\underline{t}\underline{a}, c, \underline{z}).$$

Interpretable Principles

$\mathsf{bAC}^{\rho,\tau}[\trianglelefteq]: \ \forall x^{\rho} \exists y^{\tau} A(x,y) \rightarrow \tilde{\exists} f \tilde{\forall} b \forall x \trianglelefteq b \exists y \trianglelefteq f a A(x,y),$

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Interpretable Principles

bAC$^{\rho,\tau}[\trianglelefteq]$: $\forall x^\rho \exists y^\tau A(x,y) \to \tilde{\exists} f \tilde{\forall} b \forall x \trianglelefteq b \exists y \trianglelefteq f a A(x,y)$,

bIP$^{\rho}_{\forall \mathrm{bd}}[\trianglelefteq]$: $(\forall x A_{\mathrm{b}}(x) \to \exists y^\rho B(y)) \to \tilde{\exists} b (\forall x A_{\mathrm{b}}(x) \to \exists y \trianglelefteq b B(y))$,

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$\mathsf{bMP}^{\rho}_{\mathrm{bd}}[\trianglelefteq]: \ (\forall x^\rho A_{\mathrm{b}}(x) \to B_{\mathrm{b}}) \to \tilde{\exists} a (\forall x \trianglelefteq a A_{\mathrm{b}}(x) \to B_{\mathrm{b}})$,

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$$\mathsf{bBCC}^{\rho,\tau}_{\mathrm{bd}}[\trianglelefteq] : \ \tilde{\forall} b^\tau \exists z \trianglelefteq c^\rho \forall y \trianglelefteq b A_{\mathrm{b}}(y,z) \to \exists z \trianglelefteq c \forall y A_{\mathrm{b}}(y,z),$$

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$\mathsf{MAJ}^{\rho}[\trianglelefteq]: \ \forall x^\rho \exists y (x \trianglelefteq y)$.

Soundness: First extension

- Calling all the principles above $\mathsf{P}[\trianglelefteq]$ we have:

Thm(Soundness II) If

$$\mathsf{IL}^{\omega}_{\trianglelefteq} + \mathsf{P}[\trianglelefteq] \vdash A(\underline{z}),$$

then there are closed monotone terms $\underline{t}$ of appropriate types such that

$$\mathsf{IL}^{\omega}_{\trianglelefteq} \vdash \tilde{\forall}\underline{a}\forall\underline{z} \trianglelefteq \underline{a}\,\tilde{\forall}cA_{\mathrm{B}}(\underline{ta}, c, \underline{z}).$$

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Soundness: Second extension

- Induction is interpreted using iteration functional.

Thm (Soundness III) If

$$\mathsf{HA}^{\omega}_{\unlhd} + \mathsf{P}[\unlhd] \vdash A(\underline{z}),$$

then there are closed monotone terms $\underline{t}$ of appropriate types such that

$$\mathsf{HA}^{\omega}_{\unlhd} \vdash \tilde{\forall}\underline{a}\forall\underline{z} \unlhd \underline{a}\,\tilde{\forall}c A_{\mathrm{B}}(\underline{ta}, c, \underline{z}).$$

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Interpreting Classical Theories

- $\mathsf{P}_{\mathrm{bd}}[\trianglelefteq]$: restriction of $\mathsf{P}[\trianglelefteq]$ to bounded formulas.

Thm(Negative translation) If

$$\mathsf{PA}^{\omega}_{\trianglelefteq} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash A(\underline{z}),$$

then

$$\mathsf{HA}^{\omega}_{\trianglelefteq} + \mathsf{P}_{\mathrm{bd}}[\trianglelefteq] \vdash (A(\underline{z}))^{\mathrm{N}}$$

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Uniform Weak König's Lemma

- WKL: Every infinity binary tree has an infinite path, i.e.

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Thm. $\mathsf{HA}^\omega + \mathsf{P}[\trianglelefteq] \vdash \mathsf{UWKL}$.

Example of Meta-theorem

$\mathsf{bAC}_0^{1,1} \quad : \quad \forall x^1 \exists y^1 A_0(x,y) \rightarrow \exists f \forall x \exists y \leq_1 f(x) A_0(x,y)$.

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Thm. If

$$\mathsf{PA}^\omega + \mathsf{bAC}_0^{1,1} + \mathsf{UWKL} \vdash \forall x^\rho \exists y^\tau A_0(x,y),$$

where A_0 is quantifier-free, then

$$\mathsf{PA}^\omega \vdash \tilde{\forall} a \forall x \leq^* a \exists y \leq^* q(a) A_0(x,y),$$

for some monotone closed term q.

Future work

- Feasible case

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- Bounded modified realizability

BRICS

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- Comparison with monotone functional interpretation