

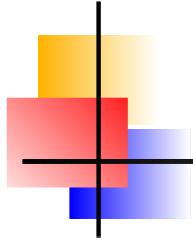
Bounded Functional Interpretation

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(joint work with F. Ferreira)

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Marseille-Luminy, January 12th - 16th, 2004

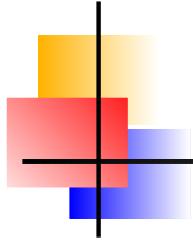


Context

- Proof mining:

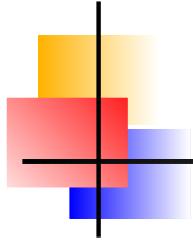
*Logical analysis of (ineffective) mathematical proofs
with the aim of extracting new information.*

- Proof mining:
Logical analysis of (ineffective) mathematical proofs with the aim of extracting new information.
- New information:
(bound on) witness for existential quantifier.



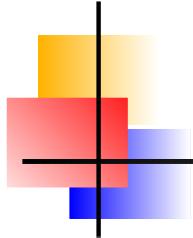
Driving Idea

If one is looking for **bounds**, then **bounded quantifiers** shouldn't have computational content.



Bounded Quantifiers

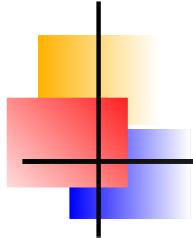
- $\forall x^{\mathbb{N}} \leq t A(x)$ intrinsically different from $\forall x^{\mathbb{N}} A(x)$.
 - Induction on NP-predicates.
 - Bounded arithmetic.



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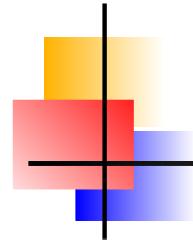
- $\forall x^{\mathbb{N} \rightarrow \mathbb{N}} \leq t A(x)$ intrinsically different from $\forall x^{\mathbb{N} \rightarrow \mathbb{N}} A(x)$.
 - $\forall x^{\mathbb{N} \rightarrow \mathbb{N}} \leq t \dots \quad \forall x \in [0, 1] \dots$
 - $\forall x^{\mathbb{N} \rightarrow \mathbb{N}} \dots \quad \forall x \in \mathbb{R} \dots$



Bounded Quantifiers

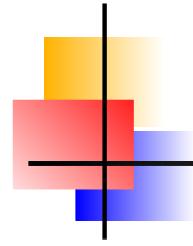
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 - $\forall x^{\mathbb{N} \rightarrow \mathbb{N}} \leq t \dots \quad \forall x \in \text{compact Polish space}$.
 - $\forall x^{\mathbb{N} \rightarrow \mathbb{N}} \dots \quad \forall x \in \text{Polish space}$.



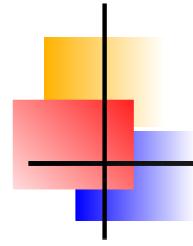
An Example

- RCA_0 : Basic theory of analysis.
- WKL : Every infinite binary tree has an infinite path.



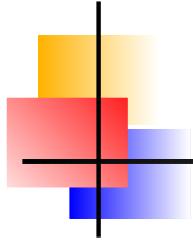
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- **Thm**(Kohlenbach'92)
If $\text{RCA}_0 + \text{WKL} \vdash \forall x \in P; y \in K_x \exists z^\tau A_\exists(x, y, z)$ then
 \exists closed term s s.t. $\forall x \in P; y \in K_x \exists z \leq s(x) A_\exists(x, y, z)$.



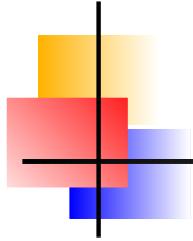
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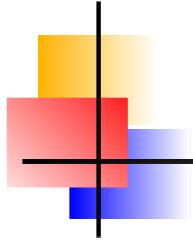
Goal

- Interpretation distinguishing $\forall x^\rho \leq t A(x)$ and $\forall x^\rho A(x)$.



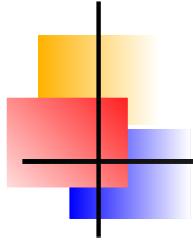
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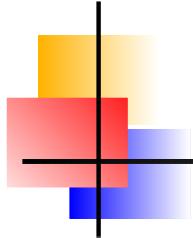
- Interpretation distinguishing $\forall x^\rho \leq t A(x)$ and $\forall x^\rho A(x)$.
- Need: bounded quantifiers for all finite types.
- Need: extended \leq to all finite types.



One Solution: Pointwise

- Use pointwise less-than-equal-to relation:

$$x \leq_{\rho \rightarrow \sigma} y := \forall z^\rho (x(z) \leq_\sigma y(z)).$$



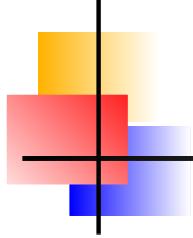
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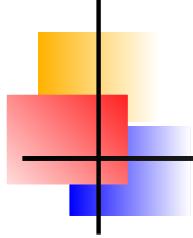
- Problem:

$x' \leq x$ and $y' \leq y$ does not imply $x'(y') \leq x(y)$.



Another Solution: Monotone

- Howard/Bezem's strong majorizability relation.
- Extension of the \leq -relation to higher types:
 - $x \leq_{\mathbb{N}}^* y := x \leq_{\mathbb{N}} y$
 - $x \leq_{\rho \rightarrow \sigma}^* y := \forall v^\rho \forall u \leq_{\rho}^* v \left(\underbrace{xu \leq_{\sigma}^* yv}_{\text{above}} \wedge \underbrace{yu \leq_{\sigma}^* yv}_{\text{monotone}} \right)$



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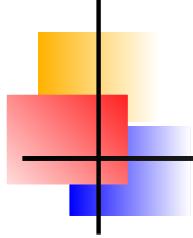
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- Example 1: for type $\mathbb{N} \rightarrow \mathbb{N}$ we have:

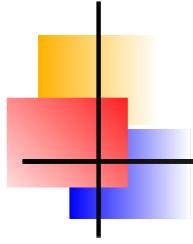
$$f \leq_{\mathbb{N} \rightarrow \mathbb{N}}^* g := \forall m \forall n \leq m \left(\underbrace{f(n) \leq g(m)}_{\text{above}} \wedge \underbrace{g(n) \leq g(m)}_{\text{monotone}} \right).$$



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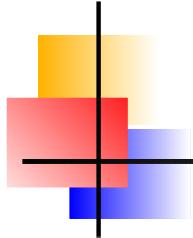
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- Example 2: $x \leq_{\rho \rightarrow \sigma}^* x$ means that x is monotone

$$x \leq_{\rho \rightarrow \sigma}^* x := \forall v^\rho \forall u \leq_\rho^* v (xu \leq_\sigma^* xv).$$



Majorizability: Some Properties

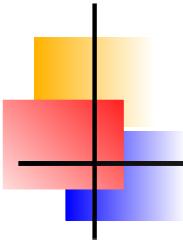
- $x \leq^* y \wedge y \leq^* z \rightarrow x \leq^* z$
- $x' \leq^* x \wedge y' \leq^* y \rightarrow x'(y') \leq^* x(y)$
- $x \leq^* y \rightarrow y \leq^* y$



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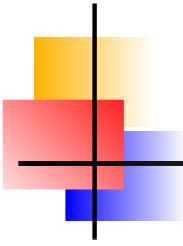
Moreover, for each closed term t of e.g. HA^ω there is another closed term t^* such that, $\text{HA}^\omega \vdash t \leq^* t^*$.



Majorizability: A New Symbol

- Idea: Add majorizability relation \trianglelefteq to the language, functional interpretation can access the relation.
- Cannot just take:
 - $x \trianglelefteq_{\mathbb{N}} y \leftrightarrow x \leq_{\mathbb{N}} y$
 - $x \trianglelefteq_{\rho \rightarrow \sigma} y \leftrightarrow \forall v^\rho \forall u^\rho \trianglelefteq_\rho v (xu \trianglelefteq_\sigma yv \wedge yu \trianglelefteq_\sigma yv)$functional interpretation would ask for a witness for

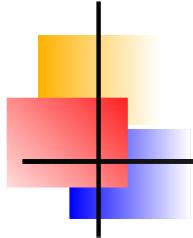
$$x \trianglelefteq_{\rho \rightarrow \sigma} y \leftarrow \forall v^\rho \forall u^\rho \trianglelefteq_\rho v (xu \trianglelefteq_\sigma yv \wedge yu \trianglelefteq_\sigma yv)$$



Majorizability: A New Symbol

- Idea: Add majorizability relation \trianglelefteq to the language, functional interpretation can access the relation.
- One solution, use a rule instead of the implication:
 - $x \trianglelefteq_{\mathbb{N}} y \leftrightarrow x \leq_{\mathbb{N}} y$
 - $x \trianglelefteq_{\rho \rightarrow \sigma} y \rightarrow \forall v^{\rho} \forall u^{\rho} \trianglelefteq_{\rho} v (xu \trianglelefteq_{\sigma} yv \wedge yu \trianglelefteq_{\sigma} yv)$

$$\frac{A_b \rightarrow \forall u \trianglelefteq v (su \trianglelefteq tv \wedge tu \trianglelefteq tv)}{A_b \rightarrow s \trianglelefteq t}$$



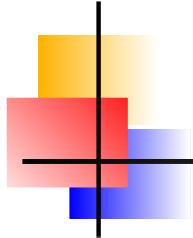
The Basic Setting

- With the intensional symbol \trianglelefteq we are in the position to define e.g. a “bounded quantifier” of arbitrary type:

$$B_{\forall} : \forall x \trianglelefteq t A(x) \leftrightarrow \forall x (x \trianglelefteq t \rightarrow A(x))$$

$$B_{\exists} : \exists x \trianglelefteq t A(x) \leftrightarrow \exists x (x \trianglelefteq t \wedge A(x)).$$

- Let the theory $IL_{\trianglelefteq}^{\omega}$ be intuitionistic logic (in all finite types) plus the axioms/rule for \trianglelefteq , B_{\forall} and B_{\exists} .

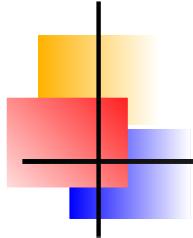


Monotone Quantifiers

- Quantify over “monotone functionals” as

$$\forall x(x \trianglelefteq x \rightarrow A(x))$$

$$\exists x(x \trianglelefteq x \wedge A(x))$$



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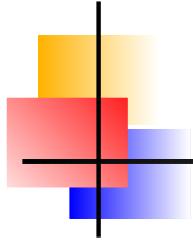
$$\forall x(x \trianglelefteq x \rightarrow A(x))$$

$$\exists x(x \trianglelefteq x \wedge A(x))$$

- Use the following abbreviations:

$\tilde{\forall}x A(x)$ instead of $\forall x(x \trianglelefteq x \rightarrow A(x))$

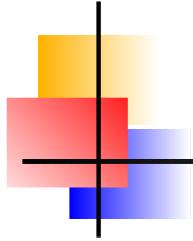
$\tilde{\exists}x A(x)$ instead of $\exists x(x \trianglelefteq x \wedge A(x))$



The Interpretation

- Main idea:

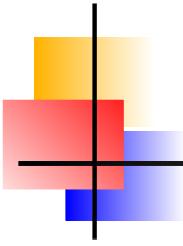
View $\forall x A(x)$ as $\tilde{\forall} b \quad \forall x \trianglelefteq b A(x)$.



The Interpretation

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View $\forall x A(x)$ as $\underbrace{\exists b}_{\text{bounding}} \overbrace{\forall x \leq b}^{\text{bounded}} A(x).$



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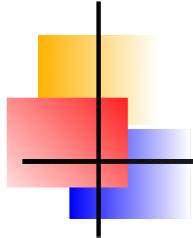
- A relativization to Bezem's model \mathcal{M} :

$$\tilde{\forall} b \forall x \leq b A(x)$$

$$\tilde{\forall} b \forall x (x \leq b \rightarrow A(x))$$

$$\forall x (\tilde{\exists} b (x \leq b) \rightarrow A(x))$$

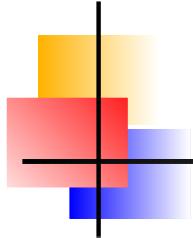
$$\forall x (x \in \mathcal{M} \rightarrow A(x))$$



The Interpretation

- Associate arbitrary formulas of $\mathcal{L}_{\trianglelefteq}^{\omega}$ to formulas having the form $\tilde{\exists}b\tilde{\forall}cA_B(b, c)$.

$$A \in \mathcal{L}_{\trianglelefteq}^{\omega} \quad \mapsto \quad (A)^B \equiv \tilde{\exists}b\tilde{\forall}cA_B(b, c).$$



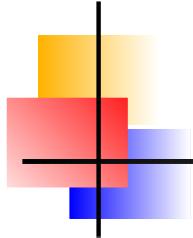
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- Compare with Gödel's functional interpretation

$$A \in \mathcal{L}^{\omega} \quad \mapsto \quad (A)^D \equiv \exists x \forall y A_{\text{qf}}(x, y).$$



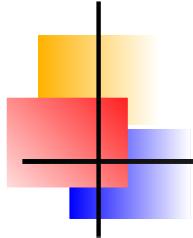
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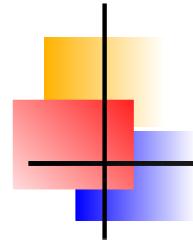
- Resulting matrix monotone on the first argument, i.e.

$$b \leq b' \wedge c \leq c \wedge A_B(b, c, x) \rightarrow A_B(b', c, x)$$



The Interpretation: Bounded quantifiers

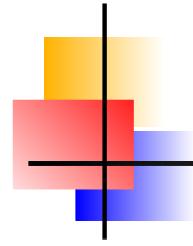
- Assume $(A(x))^B \equiv \exists b \tilde{\forall} c A_B(b, c, x)$.
- $(\forall x \trianglelefteq t A(x))^B \equiv \exists b \tilde{\forall} c \forall x \trianglelefteq t A_B(b, c, x)$.



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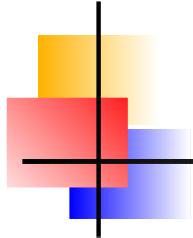
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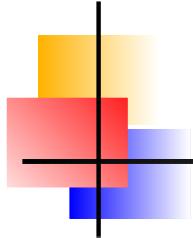
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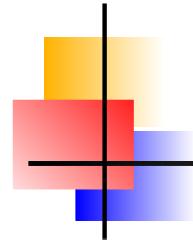
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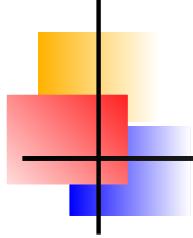
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The Interpretation: Implication

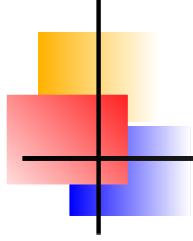
- Assume $(A)^B \equiv \exists b \forall c A_B(b, c)$ and $(B)^B \equiv \exists d \forall e B_B(d, e)$.
- $(A \rightarrow B)^B \equiv \exists f, g \forall b, e (\forall c \leq gbe A_B(b, c) \rightarrow B_B(fb, e))$



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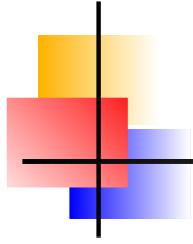
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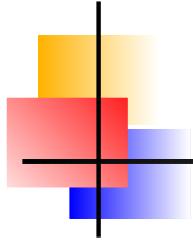
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- $(A \rightarrow B)^B \equiv \exists f, g \forall b, e (\forall c \trianglelefteq gbe A_B(b, c) \rightarrow B_B(fb, e))$

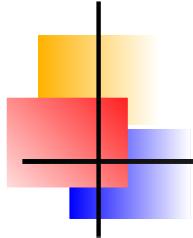
$$\begin{aligned}(A \rightarrow B)^B &\equiv \exists b \forall c A_B(b, c) \rightarrow \exists d \forall e B_B(d, e) \\ &\equiv \forall b \exists d (\forall c A_B(b, c) \rightarrow \forall e B_B(d, e)) \\ &\equiv \forall b \exists d \forall e \exists c' (\forall c \trianglelefteq c' A_B(b, c) \rightarrow B_B(d, e))\end{aligned}$$



The Interpretation: Implication

- Assume $(A)^B \equiv \exists b \forall c A_B(b, c)$ and $(B)^B \equiv \exists d \forall e B_B(d, e)$.
- $(A \rightarrow B)^B \equiv \exists f, g \forall b, e (\forall c \trianglelefteq gbe A_B(b, c) \rightarrow B_B(fb, e))$

$$\begin{aligned}(A \rightarrow B)^B &\equiv \exists b \forall c A_B(b, c) \rightarrow \exists d \forall e B_B(d, e) \\ &\equiv \forall b \exists d (\forall c A_B(b, c) \rightarrow \forall e B_B(d, e)) \\ &\equiv \forall b \exists d \forall e \exists c' (\forall c \trianglelefteq c' A_B(b, c) \rightarrow B_B(d, e)) \\ &\equiv \exists f, g \forall b, e (\forall c \trianglelefteq gbe A_B(b, c) \rightarrow B_B(fb, e)).\end{aligned}$$



The Soundness Theorem

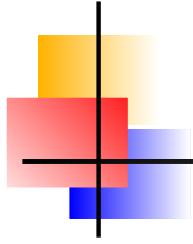
- Let $A(\underline{z})$ be an arbitrary formula of $\mathcal{L}_{\triangleleft}^{\omega}$ and assume $(A(\underline{z}))^B \equiv \exists \tilde{b} \forall \tilde{c} A_B(b, c, \underline{z})$.

Thm(Soundness I) If

$$\mathbf{IL}_{\triangleleft}^{\omega} \vdash A(\underline{z}),$$

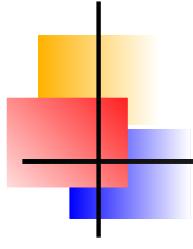
then there are closed monotone terms \underline{t} of appropriate types such that

$$\mathbf{IL}_{\triangleleft}^{\omega} \vdash \forall \underline{a} \forall \underline{z} \triangleleft \underline{a} \forall \underline{c} A_B(\underline{ta}, \underline{c}, \underline{z}).$$



Interpretable Principles

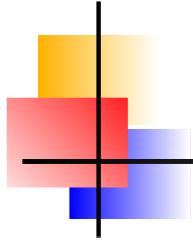
$$\text{bAC}^{\rho, \tau}[\trianglelefteq] : \forall x^\rho \exists y^\tau A(x, y) \rightarrow \tilde{\exists} f \tilde{\forall} b \forall x \trianglelefteq b \exists y \trianglelefteq f a A(x, y),$$



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$$\text{bIP}_{\forall \text{bd}}^\rho[\trianglelefteq] : (\forall x A_{\text{b}}(x) \rightarrow \exists y^\rho B(y)) \rightarrow \tilde{\exists} b (\forall x A_{\text{b}}(x) \rightarrow \exists y \trianglelefteq b B(y)),$$

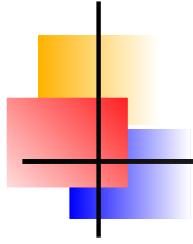


Interpretable Principles

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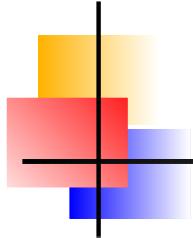
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$$\text{bBC}^{\rho, \tau}[\trianglelefteq] : \forall z \trianglelefteq c^\rho \exists y^\tau A(y, z) \rightarrow \tilde{\exists} b \forall z \trianglelefteq c \exists y \trianglelefteq b A(y, z),$$



Interpretable Principles

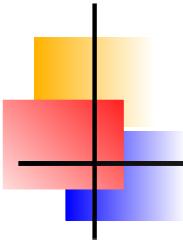
$$\text{bAC}^{\rho, \tau}[\trianglelefteq] : \forall x^\rho \exists y^\tau A(x, y) \rightarrow \tilde{\exists} f \tilde{\forall} b \forall x \trianglelefteq b \exists y \trianglelefteq f a A(x, y),$$

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$$\text{bBCC}_{\text{bd}}^{\rho, \tau}[\trianglelefteq] : \tilde{\forall} b^\tau \exists z \trianglelefteq c^\rho \forall y \trianglelefteq b A_{\text{b}}(y, z) \rightarrow \exists z \trianglelefteq c \forall y A_{\text{b}}(y, z),$$



Interpretable Principles

$$\text{bAC}^{\rho,\tau}[\trianglelefteq] : \forall x^\rho \exists y^\tau A(x, y) \rightarrow \tilde{\exists} f \tilde{\forall} b \forall x \trianglelefteq b \exists y \trianglelefteq f a A(x, y),$$

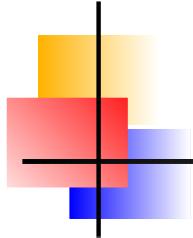
$$\text{bIP}_{\forall \text{bd}}^\rho[\trianglelefteq] : (\forall x A_{\text{b}}(x) \rightarrow \exists y^\rho B(y)) \rightarrow \tilde{\exists} b (\forall x A_{\text{b}}(x) \rightarrow \exists y \trianglelefteq b B(y)),$$

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$$\text{bBCC}_{\text{bd}}^{\rho,\tau}[\trianglelefteq] : \tilde{\forall} b^\tau \exists z \trianglelefteq c^\rho \forall y \trianglelefteq b A_{\text{b}}(y, z) \rightarrow \exists z \trianglelefteq c \forall y A_{\text{b}}(y, z),$$

$$\text{MAJ}^\rho[\trianglelefteq] : \forall x^\rho \exists y (x \trianglelefteq y).$$



Soundness: First extension

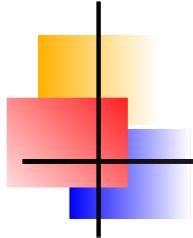
- Calling all the principles above $P[\triangleleft]$ we have:

Thm(Soundness II) If

$$\mathbf{IL}_{\triangleleft}^{\omega} + P[\triangleleft] \vdash A(\underline{z}),$$

then there are closed monotone terms \underline{t} of appropriate types such that

$$\mathbf{IL}_{\triangleleft}^{\omega} \vdash \tilde{\forall} \underline{a} \forall \underline{z} \trianglelefteq \underline{a} \tilde{\forall} \underline{c} A_B(\underline{ta}, \underline{c}, \underline{z}).$$



Soundness: Second extension

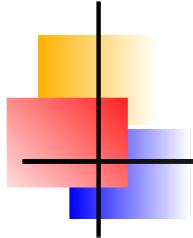
- Induction is interpreted using iteration functional.

Thm(Soundness III) If

$$\text{HA}_{\trianglelefteq}^{\omega} + \text{P}[\trianglelefteq] \vdash A(\underline{z}),$$

then there are closed monotone terms \underline{t} of appropriate types such that

$$\text{HA}_{\trianglelefteq}^{\omega} \vdash \tilde{\forall} \underline{a} \forall \underline{z} \trianglelefteq \underline{a} \tilde{\forall} \underline{c} A_B(\underline{ta}, \underline{c}, \underline{z}).$$



Interpreting Classical Theories

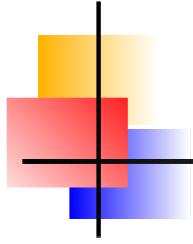
- $P_{bd}[\trianglelefteq]$: restriction of $P[\trianglelefteq]$ to bounded formulas.

Thm(Negative translation) If

$$PA_{\trianglelefteq}^\omega + P_{bd}[\trianglelefteq] \vdash A(\underline{z}),$$

then

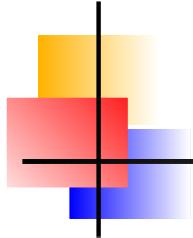
$$HA_{\trianglelefteq}^\omega + P_{bd}[\trianglelefteq] \vdash (A(\underline{z}))^N$$



Uniform Weak König's Lemma

- WKL: Every infinity binary tree has an infinite path, i.e.

$$\forall T(\text{Inf}(T) \wedge \text{Bin}(T) \rightarrow \exists p(\text{Inf}(p) \wedge p \in T)).$$



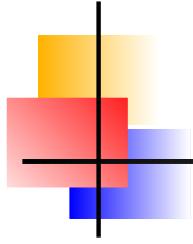
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- UWKL: Uniform version of weak König's lemma:

$$\exists \Phi \forall T(\text{Inf}(T) \wedge \text{Bin}(T) \rightarrow (\text{Inf}(\Phi(T)) \wedge \Phi(T) \in T)).$$



Uniform Weak König's Lemma

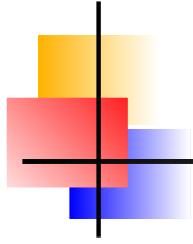
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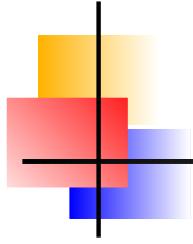
$$\exists \Phi \forall T(\text{Inf}(T) \wedge \text{Bin}(T) \rightarrow (\text{Inf}(\Phi(T)) \wedge \Phi(T) \in T)).$$

Thm. $\text{HA}^\omega + \text{P}[\trianglelefteq] \vdash \text{UWKL}$.



Example of Meta-theorem

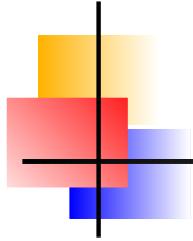
$$\text{bAC}_0^{1,1} : \forall x^1 \exists y^1 A_0(x, y) \rightarrow \exists f \forall x \exists y \leq_1 f(x) A_0(x, y).$$



Example of Meta-theorem

$$\text{bAC}_0^{1,1} : \forall x^1 \exists y^1 A_0(x, y) \rightarrow \exists f \forall x \exists y \leq_1 f(x) A_0(x, y).$$

Lemma. $\text{PA}_{\triangleleft}^{\omega} \vdash A(z) \Rightarrow \text{PA}^{\omega} \vdash A(z)[\leq^* / \triangleleft].$



Example of Meta-theorem

$$\text{bAC}_0^{1,1} : \forall x^1 \exists y^1 A_0(x, y) \rightarrow \exists f \forall x \exists y \leq_1 f(x) A_0(x, y).$$

Lemma. $\text{PA}_{\triangleleft}^{\omega} \vdash A(z) \Rightarrow \text{PA}^{\omega} \vdash A(z)[\leq^* / \triangleleft].$

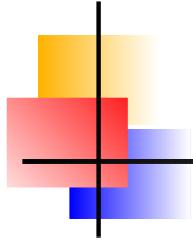
Thm. If

$$\text{PA}^{\omega} + \text{bAC}_0^{1,1} + \text{UWKL} \vdash \forall x^{\rho} \exists y^{\tau} A_0(x, y),$$

where A_0 is quantifier-free, then

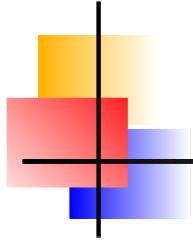
$$\text{PA}^{\omega} \vdash \tilde{\forall} a \forall x \leq^* a \exists y \leq^* q(a) A_0(x, y),$$

for some monotone closed term q .



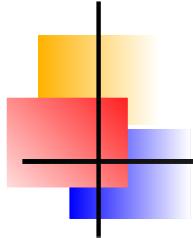
Future work

- Feasible case



Future work

- Feasible case
- Bounded modified realizability



Future work

- Feasible case
- Bounded modified realizability
- Comparison with monotone functional interpretation