Polynomial-time programs from ineffective proofs in feasible analysis

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The Plan

1. Motivation
   - Ineffective principles in analysis (weak König's Lemma)
   - Feasible analysis
2. The Main Result
   - Algorithm for extracting polynomial-time realizers from proofs (involving WKL) of Π^0_2-theorems in feasible analysis.
3. Sketch of the Proof
4. Related/Future Work

Ineffective principles

- By ineffective principles we mean, e.g.

(1) Heine/Borel covering lemma for $[0, 1]$,

(2) Every continuous function $f : [0, 1] \to \mathbb{R}$ attains its infimum and supremum,

(3) Every continuous function $f : [0, 1] \to \mathbb{R}$ is uniformly continuous.

- Over a basic system of analysis (RCA_0) those principles are equivalent to

WKL : *Every infinite binary tree has an infinite branch*

- This principle is normally called binary/weak König's Lemma.

- WKL is ineffective in the sense that it only holds in models which contain non-recursive functions.

WKL in proofs of $\forall\exists$-theorems

- What if WKL is used in the proof of a theorem $\forall x \exists y A_0(x, y)$?
- In 76 Friedman defined the subsystem of analysis RCA_0 and showed that RCA_0 is Π^0_2-conservative over PRA, i.e.

  Thm [Friedman]. If $\mathsf{RCA}_0 \vdash \forall x \exists y A_0(x, y)$ then there exists a primitive recursive function f such that $\mathsf{PRA} \vdash A_0(x, fx)$.

- Moreover, he showed that $\mathsf{RCA}_0 + \mathsf{WKL}$ is Π^0_2-conservative over RCA_0. Therefore:

  Thm [Friedman]. If $\mathsf{RCA}_0 + \mathsf{WKL} \vdash \forall x \exists y A_0(x, y)$ then there exists primitive recursive function f such that $\mathsf{PRA} \vdash A_0(x, fx)$.

- Friedman's proof is ineffective!

On Friedman's result

- Harrington'77 proved (also non-constructively) Π^1_1-conservation of WKL over RCA_0.
- First effective version of Friedman's result was given by Sieg'85 (based on cut-elimination).
- Extension of Friedman's result to the higher types was given by Kohlenbach'92 (based on functional interpretation).
- Avigad'96 formalized the forcing argument used in Harrington's proof obtaining an effective version of the Π^1_1-conservation result (no function extraction procedure, though)

Basic Feasible Analysis I

- Ferreira'94 defined a Basic Theory for Feasible Analysis BTFA
- The Π^0_2-theorems of BTFA have polynomial-time computable realizers.

  Thm [Ferreira]. If $\mathsf{BTFA} \vdash \forall x \exists y A_0(x, y)$ then there exists a polynomial-time computable function f such that $\forall x A_0(x, fx)$ holds.

- Ferreira also showed non-constructively that BTFA and $\mathsf{BTFA} + \mathsf{WKL}$ have the same Π^0_2-theorems. Hence:

  Thm [Ferreira]. If $\mathsf{BTFA} + \mathsf{WKL} \vdash \forall x \exists y A_0(x, y)$ then there exists a polynomial-time computable function f such that $\forall x A_0(x, fx)$ holds.

Basic Feasible Analysis II

- A different basic theory for feasible analysis (based on the language of finite types) can be obtained by taking Cook and Urquhart's system CPV^ω extended with quantifier-free choice $\mathsf{QF\text{-}AC}$.

- The resulting theory can be viewed as an extension of (a version of) BTFA to all finite types.

  Thm. If $\mathsf{CPV}^\omega + \mathsf{QF\text{-}AC} \vdash \forall x \exists y A_0(x, y)$ then there exists *effectively* a polynomial-time computable function f such that $\mathsf{IPV}^\omega \vdash \forall x A_0(x, fx)$.

Main result

Thm. If $\mathsf{CPV}^\omega + \mathsf{QF\text{-}AC} + \mathsf{WKL} \vdash \forall x \exists y A_0(x,y)$ then there exists *effectively* a polynomial-time computable function f such that $\forall x A_0(x, fx)$ holds.

- We can also allow "set parameters" in the theorem above, i.e.

  Thm. If $\mathsf{CPV}^\omega + \mathsf{QF\text{-}AC} + \mathsf{WKL} \vdash \forall x \exists y A_0(x,y,\alpha)$ then there exists *effectively* a polynomial-time computable function *with boolean oracle* f such that $\forall x \forall \alpha : \{0,1\}^\omega A_0(x, fx\alpha, \alpha)$ holds.

- In order to illustrated the mathematical significance of the system $\mathsf{CPV}^\omega + \mathsf{QF\text{-}AC} + \mathsf{WKL}$ we have indicated how to formalize the proof of Heine/Borel covering lemma in it.

Sketch of the proof

1. Cook and Urquhart showed that CPV^ω has a functional interpretation, via negative translation, in IPV^ω.

   Thm [CU'93]. $\mathsf{CPV}^\omega \xrightarrow{\mathsf{N+f.i.}} \mathsf{IPV}^\omega$.

2. We extend this interpretation to $\mathsf{CPV}^\omega + \mathsf{QF\text{-}AC}$.

   Lem. $\mathsf{CPV}^\omega + \mathsf{QF\text{-}AC} \xrightarrow{\mathsf{N+f.i.}} \mathsf{IPV}^\omega$.

3. And, by adding a new form of binary bar recursion $\mathcal{B}$ to IPV^ω we can even interpret WKL.

   Thm. $\mathsf{CPV}^\omega + \mathsf{QF\text{-}AC} + \mathsf{WKL} \xrightarrow{\mathsf{N+f.i.}} \mathsf{IPV}^\omega + \mathcal{B}$.

4. Finally, we show that the functions of $\mathsf{IPV}^\omega + \mathcal{B}$ are polynomial-time computable.

   Thm. $[\mathsf{IPV}^\omega + \mathcal{B}]_1 \equiv \mathbf{P}$.

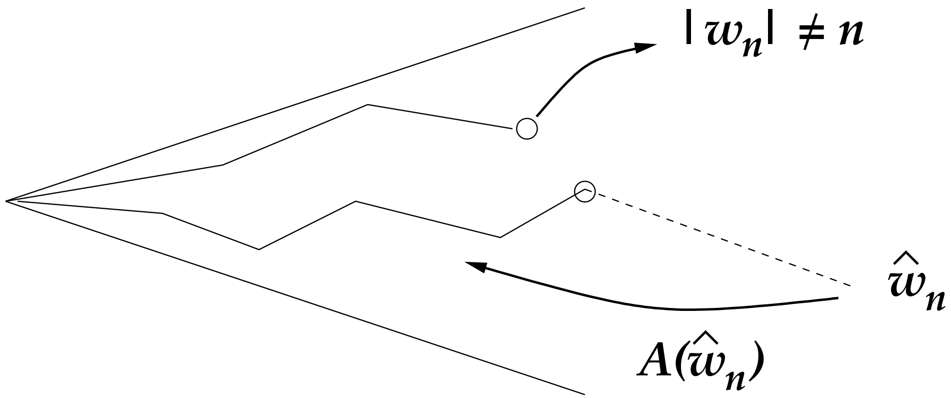
The Functional Interpretation of WKL

$$\hat{w}_n := w_n * 0000\ldots \qquad \overline{\alpha}k := \alpha(0)\alpha(1)\ldots\alpha(k-1)$$

Problem: Given a binary tree T, a function $A : \{0,1\}^\omega \to \mathbb{N}$ and a sequence of finite branches $(w_i)_{i\in\mathbb{N}}$, produce n and $\alpha : \{0,1\}^\omega$ satisfying:

$$|w_n| = n \wedge T(w_n) \to T(\overline{\alpha}(A\alpha)).$$

- Two possible solutions:



Binary Bar Recursion

$$\mathcal{B}(A,(w_i)_{i\in\mathbb{N}},n) = \begin{cases} n & \text{if } |A\hat{w}_n| \le |w_n| \\ & \text{or } |w_n| \neq n \\ \mathcal{B}(A,(w_i)_{i\in\mathbb{N}},n+1) & \text{otherwise,} \end{cases}$$

where $A : \{0,1\}^\omega \to \mathbb{N}$ and $w_i : \{0,1\}^*$.

- It can be also formulated in the form of an unbounded search:

  $\min m \ge n\,(|A\hat{w}_m| \le |w_m| \vee |w_m| \neq m)$

- How to justify such recursion?

  Lem [KC'96]. For any closed term $\Psi : \mathbb{N} \to \{0,1\}^\omega \to \mathbb{N}$ of IPV^ω, there exist constants c_1 and c_2 such that

$$\forall x : \mathbb{N}\forall\alpha : \{0,1\}^\omega(|\Psi x\alpha| \le |x|^{c_1} + c_2)$$

Eliminating the Bar Recursion I

- Suppose we have type one term $t : \mathbb{N} \to \mathbb{N}$ in the language of $\mathsf{IPV}^\omega + \mathcal{B}$, we show how to replace $\mathcal{B}$ by limited recursion on notation.
- In fact, for the induction hypothesis we need a stronger condition:

Lem. For any term $t[x, \alpha] : \mathbb{N}$ of $\mathsf{IPV}^\omega + \mathcal{B}$, there exists a term $t'[x, \alpha] : \mathbb{N}$ of IPV^ω such that

$$\forall x : \mathbb{N} \forall \alpha : \{0,1\}^\omega (t[x, \alpha] = t'[x, \alpha]).$$

Eliminating the Bar Recursion II

- Let $\Psi[x,\alpha]$ and $(w_i)_{i\in\mathbb{N}}[x,\alpha]$ be fixed terms of $\mathsf{IPV}^\omega + \mathcal{B}$. The main step is to show that $\mathcal{B}(\Psi[x,\alpha],(w_i)_{i\in\mathbb{N}}[x,\alpha],0)$, i.e. (omitting $[x,\alpha]$)

$$\min n\,(|\Psi\hat{w}_n| \leq |w_n| \vee |w_n| \neq n),$$

  can be replaced by limited recursion on notation.

- This can be done since $|\Psi x\alpha|$, and hence the search, is bounded by $|x|^{c_1} + c_2$.

- Therefore, given an arbitrary term $t[x,\alpha]$ in $\mathsf{IPV}^\omega + \mathcal{B}$, we can successively normalize it and replace the innermost occurrence of $\mathcal{B}$ by limited recursion on notation.

Related Work

- Howard'81 used a different form of binary bar recursion to realize the functional interpretation of (the negative translation of) WKL.
- Howard's binary bar recursion, however, seems to be too strong for the feasible context, since it apparently involves an exponential search.
- Sieg's proof of WKL-elimination (based on cut elimination) was successfully adapted to the feasible setting by Kauffmann'00.
- Our approach directly extracts a polynomial-time computable realizer out of the WKL-proof, rather than eliminating it first.

Future Work

- Investigate whether Kohlenbach's effective proofs of WKL elimination can be translated into the feasible setting, by making a careful treatment of bounded quantifiers.
- Find ineffective proofs of Π^0_2-theorems which can be formalized in $\mathsf{CPV}^\omega + \mathsf{QF\text{-}AC} + \mathsf{WKL}$, and carry out the extraction of polynomial-time algorithms (cf. analysis of WKL-proofs e.g. in approximation theory).
- Compare the quality of the polynomial-time algorithms yielded via the approach based on cut elimination and our approach.