# On modified bar recursion

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# The Plan

1. Bar recursion:
   - Spector's bar recursion, (SBR)
   - Kohlenbach's bar recursion, (KBR)
   - Modified bar recursion. (MBR)
2. Previous results concerning MBR
   - $\mathcal{M}$ (majorizable functionals) satisfies MBR.
   - MBR is not S1-S9 computable in the total functionals.
   - MBR defines SBR.
3. New results:
   - The functional $\Gamma$ (Gandy/Hyland)
   - A 'weak' form of modified bar recursion

# Spector's bar recursion

- Spector's extension of Gödel's **T** consists of adding new functional symbols $\mathsf{SBR}_{\rho,\tau}$ to the language, and the axiom schema

$$\mathsf{SBR}_{\rho,\tau}(Y, G, H, s) \stackrel{\tau}{=} \begin{cases} G(s) & Y(s * \mathbf{0}) \stackrel{\mathbb{N}}{<} |s| \\ H(s, \lambda x^\rho.\mathsf{SBR}(s * x)) & \text{otherwise.} \end{cases}$$

  **Thm**[Spector'62] From a proof

$$\mathbf{PA}^\omega + \mathbf{AC}_0 \vdash \forall x^\sigma \exists y^\rho \; A_{qf}(x, y),$$

  one can extract a closed term $t^{\sigma \to \rho}$ (of $\mathbf{HA}^\omega + \mathsf{SBR}$) such that

$$\mathbf{HA}^\omega + \mathsf{SBR} \vdash A_{qf}(x, tx).$$

- Bezem'86 showed that $\mathcal{M}$ (the type structure of strongly majorizable functionals) is a model of $\mathsf{SBR}$.

## Kohlenbach's bar recursion

- In Kohlenbach'90 the following bar recursive functional is defined:

$$\mathsf{KBR}_{\rho,\tau}(Y,G,H,s) \stackrel{\tau}{=} \begin{cases} G(s) & Y(s*\mathbf{0}) \stackrel{\mathbb{N}}{=} Y(s*\mathbf{1}) \\ H(s,\lambda x^{\rho}.\mathsf{KBR}(s*x)) & \text{otherwise.} \end{cases}$$

- KBR defines SBR primitive recursively.
- The functional KBR is not majorizable, i.e. $\mathcal{M}$ is not a model of KBR. Hence, KBR is not primitive recursively definable in SBR.
- KBR is primitive recursively definable in SBR plus

$$\mu(Y,\alpha^{\mathbb{N}^\omega},k) :\equiv \min n \geq k\ [Y(\overline{\alpha}n*\mathbf{0}) = Y(\overline{\alpha}n*\mathbf{1})].$$

# Modified bar recursion

- Modified bar recursion is defined as

$$\text{MBR}(Y, H, s) \stackrel{\mathbb{N}}{=} Y(s * \underbrace{H(s, \lambda x^\rho.\text{MBR}(Y, H, s * x))}_{\rho^\omega}).$$

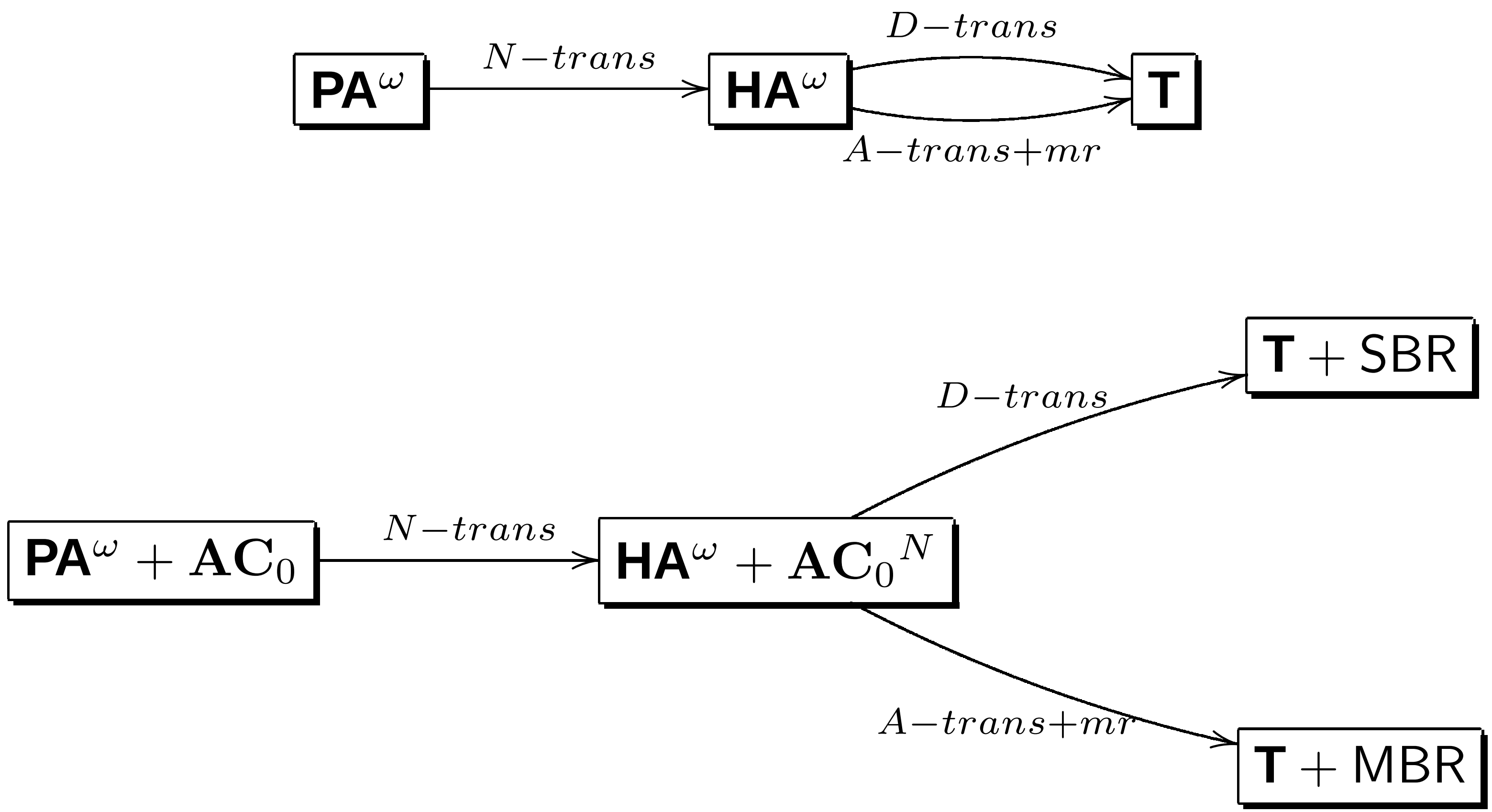
**Thm**[BO'02] From a proof

$$\text{PA}^\omega + \mathbf{AC}_0 \vdash \forall x^\sigma \, \exists y^{\mathbb{N}} \, A_{at}(x, y)$$

where $A_{at}(x, y)$ is atomic, one can extract a closed term $t^{\sigma \to \mathbb{N}}$ (of $\textbf{HA}^\omega + \text{MBR}$) such that

$$\textbf{HA}^\omega + \textbf{Continuity} + \text{MBR} \vdash A_{qf}(x, tx).$$

# Interpreting Arithmetic and Analysis



# The model of strongly majorizable functionals

**Thm**[BO'02] $\mathcal{M}$ is a model of modified bar recursion.

- We need two steps:

1. There exists a functional

   $\Phi \in \mathcal{M}_{\rho^\omega \to \mathbb{N}} \times \mathcal{M}_{\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega} \times \mathcal{M}_{\rho^*} \to \mathcal{M}_{\mathbb{N}}$

   satisfying the equation MBR.

2. There exists a functional $\Phi^*$ majorizing $\Phi$, then

   $\Phi \in \mathcal{M}_{(\rho^\omega \to \mathbb{N}) \times (\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega) \times \rho^* \to \mathbb{N}}$

## Fan functional

- A continuous functional $Y : \mathbb{N}^\omega \to \mathbb{N}$ on the Cantor subspace ($\{0,1\}^\omega$) is uniformly continuous.
- A fan functional $\Phi : (\{0,1\}^\omega \to \mathbb{N}) \to \mathbb{N}$ on input $Y : \{0,1\}^\omega \to \mathbb{N}$ should produce a point of uniform continuity $n$, i.e.

$$\forall \alpha, \beta \in \{0,1\}^\omega \; (\overline{\alpha}n = \overline{\beta}n \to Y(\alpha) = Y(\beta))$$

**Thm**[Tait] No fan functional is S1-S9 computable over $\mathcal{C}$ (the model of total continuous functionals).

[Berger] However, the minimal fan functional is S1-S9 computable over $\hat{\mathcal{C}}$ (the model of partial continuous functionals).

## Defining the fan functional

- Using MBR one defines

$$\Phi(s,v) \stackrel{\mathbb{N}^\omega}{=} s \,@\, [\text{if } Y(\Phi(s*0,v)) \neq v \text{ then } \Phi(s*0,v)$$
$$\text{else } \Phi(s*1,v)]$$

- Using KBR (and the functional $\Phi$) one defines

$$\Psi_Y(s) = \begin{cases} 0 & \text{if } Y(\hat{s}) \stackrel{\mathbb{N}}{=} Y(\Phi(s,v)) \\ 1 + \max\{\Psi_Y(s*0), \Psi_Y(s*1)\} & \text{otherwise,} \end{cases}$$

where $\hat{s} = s * \mathbf{0}$ and $v = Y(s * \mathbf{0})$.

- $\lambda Y.\Psi_Y(\langle\,\rangle)$ is a fan functional. Therefore, MBR is not S1-S9 computable in $\mathcal{C}$.

- Moreover, since Spector's bar recursion is S1-S9 computable in $\mathcal{C}$ we can conclude:

  **Thm**[BO'02] SBR does not define (primitive recursively) modified bar recursion.

# Fan functional (cont.)

- FAN (the minimal fan functional) is primitive recursively definable in MBR + KBR.
- Note that:
  - FAN is not S1-S9 computable in $\mathcal{C}$ and it is not majorizable.
  - MBR is not S1-S9 computable in $\mathcal{C}$ but it is majorizable.
  - KBR is S1-S9 computable in $\mathcal{C}$ but it is not majorizable.

# Using MBR to define SBR

**Lemma**[BO'02] Modified bar recursion can also be used to define (primitive recursively) the following search functional

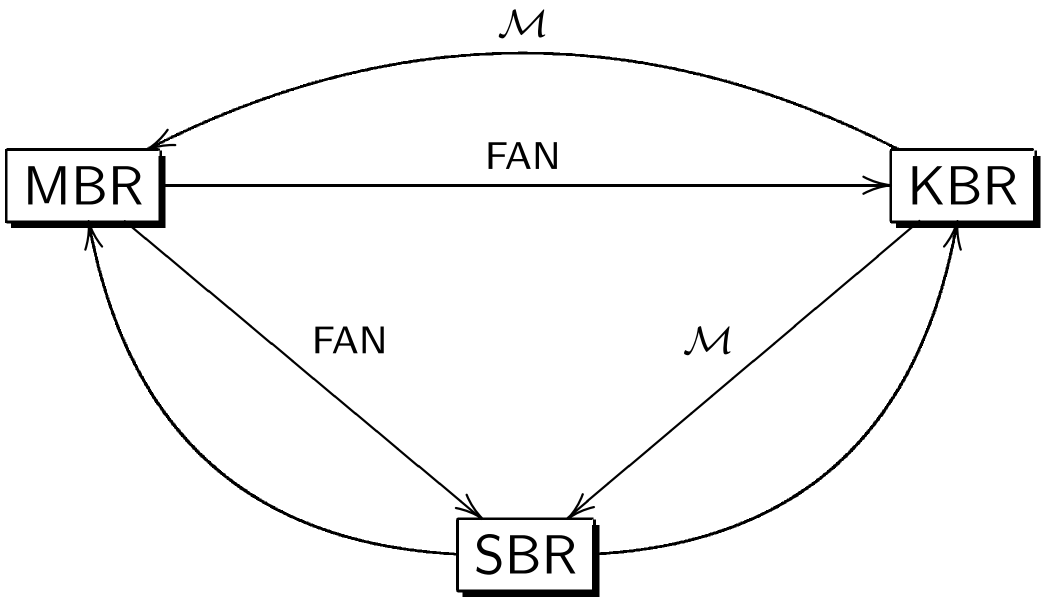
$$\tilde{\mu}(Y, \alpha^{\rho^\omega}, k) :\equiv \min n \geq k\, [Y(\overline{\alpha}n * \mathbf{0}) < n].$$

**Lemma**[BO'02] $\mathsf{SBR}_{\rho,\mathbb{N}}$ is primitive recursively definable in $\tilde{\mu} + \mathsf{MBR}_\rho$ $(= \mathsf{MBR}_\rho)$

**Lemma**[BO'02] $\mathsf{SBR}_{\rho,\tau}$ is primitive recursively definable in $\mathsf{SBR}_{\rho',\mathbb{N}}$, where if $\tau = \tau_1 \to \ldots \to \tau_n \to \mathbb{N}$ then $\rho' = \rho \times \tau_1 \times \ldots \times \tau_n$.

**Thm**[BO'02] SBR is primitive recursively definable in MBR.

# Summary of previous results



## The functional $\Gamma$

- The functional $\Gamma$ [Gandy/Hyland] is defined as follows
  $\Gamma(Y, s) \stackrel{\mathbb{N}}{=} Y(s * 0 * \lambda n^{\mathbb{N}}.\Gamma(Y, s * (n + 1)))$.
- The functional $\Gamma$ has a recursive associate but is not S1-S9 computable in the model $\mathcal{C}$ of total continuous functionals (even in the fan functional).

  **Theorem.** The functional $\Gamma$ is primitive recursively equivalent to $\mathrm{MBR}_0$.

  **Corollary.** $\mathrm{MBR}_0$ is not S1-S9 computable in the model $\mathcal{C}$ of total continuous functionals, not even in the fan functional.

# Weak modified bar recursion

- As shown in [BO'02] only a weak form of modified bar recursion is necessary for realizing (the negative translation of) countable and dependent choice, namely

$$\mathsf{wMBR}(Y, H, s) \stackrel{\mathbb{N}}{=} Y(s * \lambda k.\, \underbrace{H(s, \lambda x^{\rho}.\mathsf{wMBR}(Y, H, s * x))}_{\rho}).$$

**Theorem.** $\mathsf{wMBR}_{\rho}$ defines $\mathsf{MBR}_{\rho}$ primitive recursively for $\rho > 0$.

- It is still open whether $\mathsf{wMBR}_0$ also defines $\mathsf{MBR}_0$ primitive recursively.

## New positive results