Modified bar recursion

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The Plan

1. Bar recursion:
   - Spector’s bar recursion, \((SBR)\)
   - Kohlenbach’s bar recursion, \((KBR)\)
   - Modified bar recursion. \((MBR)\)

2. Interpreting classical Analysis \((\mathbf{PA}^\omega + \mathbf{AC}_0)\) in \(T + MBR\).

3. The model \(\mathcal{M}\) and \(MBR\).

4. Using \(MBR + KBR\) to define the fan functional.
   - S1-S9 computability.

5. Using \(MBR\) to define \(SBR\).
Spector’s bar recursion

- Spector’s extension of Gödel’s T consists of adding new functional symbols $SBR_{\rho,\tau}$ to the language, and the axiom schema

$$SBR_{\rho,\tau}(Y, G, H, s) \triangleq \begin{cases} G(s) & Y(s \cdot 0) \leq |s| \\ H(s, \lambda x^\rho. SBR(s \cdot x)) & \text{otherwise.} \end{cases}$$

**Thm**[Spector’62] From a proof

\[ \text{PA}^\omega + \text{AC}_0 \vdash \forall x^\sigma \exists y^\rho A_{qf}(x, y), \]

one can extract a closed term $t^{\sigma \rightarrow \rho}$ (of $\text{HA}^\omega + SBR$) such that

\[ \text{HA}^\omega + SBR \vdash A_{qf}(x, tx). \]

- Bezem’86 showed that $\mathcal{M}$ (the type structure of strongly majorizable functionals) is a model of $SBR$. 

Modified bar recursion
In Kohlenbach'90 the following bar recursive functional is defined:

\[ K_{BR}^{\rho,\tau}(Y, G, H, s) \equiv \begin{cases} 
G(s) & Y(s \ast 0) = Y(s \ast 1) \\
H(s, \lambda x^\rho.K_{BR}(s \ast x)) & \text{otherwise.}
\end{cases} \]

- **KBR** defines SBR primitive recursively.

- The functional **KBR** is not majorizable, i.e. \( \mathcal{M} \) is not a model of KBR. Hence, KBR is not primitive recursively definable in SBR.

- **KBR** is primitive recursively definable in SBR plus

\[ \mu(Y, \alpha^{N^\omega}, k) \equiv \min n \geq k [Y(\overline{\alpha}n \ast 0) = Y(\overline{\alpha}n \ast 1)]. \]
Modified bar recursion

- Modified bar recursion is defined as

\[\text{MBR}(Y, H, s) \equiv Y(s \ast H(s, \lambda x.\text{MBR}(Y, H, s \ast x)))).\]

**Thm**[BO’02] From a proof

\[\text{PA}^\omega + \text{AC}_0 \vdash \forall x^\sigma \exists y^\mathbb{N} A_{qf}(z, y)\]

where \(A_{qf}(x, y)\) is quantifier free, one can extract a closed term \(t^\sigma \rightarrow^\mathbb{N}\) (of \(\text{HA}^\omega + \text{MBR}\)) such that

\[\text{HA}^\omega + \text{Continuity} + \text{MBR} \vdash A_{qf}(x, tx).\]
Interpreting Arithmetic and Analysis

\[
\text{PA}^\omega \quad \xrightarrow{N\text{-}trans} \quad \text{HA}^\omega \quad \xrightarrow{A\text{-}trans+mr} \quad T
\]

\[
\text{PA}^\omega + AC_0 \quad \xrightarrow{N\text{-}trans} \quad \text{HA}^\omega + AC_0^N \quad \xrightarrow{A\text{-}trans+mr} \quad T + MBR
\]

\[
\text{HA}^\omega \quad \xrightarrow{D\text{-}trans} \quad T + SBR
\]

Modified bar recursion
**The model of strongly majorizable functionals**

**Thm [BO’02]** \( M \) is a model of modified bar recursion.

- We need two steps:
  1. There exists a functional
     \[
     \Phi \in M_{\rho^\omega \to \mathbb{N}} \times M_{\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega} \times M_{\rho^* \to \mathbb{N}}
     \]
     satisfying the equation \( MBR \).
  2. There exists a functional \( \Phi^* \) majorizing \( \Phi \), then
     \[
     \Phi \in M_{(\rho^\omega \to \mathbb{N}) \times (\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega) \times \rho^* \to \mathbb{N}}
     \]
We first prove a “weak continuity property” of the majorizable functionals of type \( \rho^\omega \to \mathbb{N} \).

**Lemma** [BO’02] The following is true in \( \mathcal{M} \),
\[
\forall Y^{\rho^\omega \to \mathbb{N}}, \alpha^{\rho^\omega} \exists n^\mathbb{N} \forall \beta^{\rho^\omega} (Y(\overline{\alpha}n \ast \beta) < n).
\]

- Let \( Y^{\rho^\omega \to \mathbb{N}} \) and \( H^{\rho^\omega \times (\rho \to \mathbb{N}) \to \rho^\omega} \) (in \( \mathcal{M} \)) be fixed. We show that a \( \Phi \) exists such that, for all \( s \),
\[
\text{(\ast)} \quad \Phi(s) = Y(s \odot H(s, \lambda x. \Phi(s \ast x))).
\]
- For a fixed \( \alpha \) pick \( n \) as in the lemma above. Then for \( s \) extending \( \overline{\alpha}n \) we must have \( \Phi(s) \in \{0, \ldots, n\} \).
- By a fixed point argument we can show that a solution for equation (\ast) exists for all \( s \) extending \( \overline{\alpha}n \). By bar induction we get a solution for all \( s \).
A functional $\Omega : \rho \to \rho$ is defined in Kohlenbach’90 such that,

i) For all $F$, $\Omega(F)$ majorizes $F$,

ii) $\Omega \in M$.

By the weak continuity property of $M$ we also get that the functional

$$\Gamma(Y)(\alpha) := \min n \left[ \forall \beta \in \alpha \exists n(\Omega(Y)(\beta) \leq n) \right]$$

is well-defined. Moreover, $\Gamma \in M$.

We can then show that

$$\Phi^* := \lambda Y, H. \Omega(\Phi \hat{Y} \hat{H})$$

majorizes $\Phi$, where $\hat{Y}(\alpha) := \Gamma(Y)(\Omega(\alpha))$ and $\hat{H}(s, \alpha) := H(\Omega(s), \Omega(\alpha))$. 

**MBR in $M$ (step 2.)**
Fan functional

- A continuous functional \( Y : \mathbb{N}^\omega \to \mathbb{N} \) on the Cantor subspace \((\{0, 1\}^\omega)\) is uniformly continuous.

- A fan functional \( \Phi : (\{0, 1\}^\omega \to \mathbb{N}) \to \mathbb{N} \) on input \( Y : \{0, 1\}^\omega \to \mathbb{N} \) should produce a point of uniform continuity \( n \), i.e.

\[
\forall \alpha, \beta \in \{0, 1\}^\omega \ (\bar{\alpha}n = \bar{\beta}n \to Y(\alpha) = Y(\beta))
\]

Thm[Tait] No fan functional is S1-S9 computable over \( \mathcal{C} \) (the model of total continuous functionals).

[Berger] However, the minimal fan functional is S1-S9 computable over \( \hat{\mathcal{C}} \) (the model of partial continuous functionals).
Defining the fan functional

- Using MBR one defines
  \[ \Phi(s, v) = s \oplus \begin{cases} 
  Y(\Phi(s \ast 0, v)) \neq v & \text{if } Y(\Phi(s \ast 0, v)) \neq v \\
  \Phi(s \ast 1, v) & \text{else}
  \end{cases} \]

- Using KBR (and the functional \( \Phi \)) one defines
  \[ \Psi_Y(s) = \begin{cases} 
  0 & \text{if } Y(\hat{s}) = \mathbb{N} \quad Y(\Phi(s, v)) \\
  1 + \max\{\Psi_Y(s \ast 0), \Psi_Y(s \ast 1)\} & \text{otherwise,}
  \end{cases} \]

where \( \hat{s} = s \ast 0 \) and \( v = Y(s \ast 0) \).

- \( \lambda Y. \Psi_Y(\langle \rangle) \) is a fan functional. Therefore, MBR is not S1-S9 computable in \( C \).

- Moreover, since Spector’s bar recursion is S1-S9 computable in \( C \) we can conclude:

  **Thm**[BO’02] SBR does not define (primitive recursively) modified bar recursion.
Fan functional (cont.)

- **FAN** (the minimal fan functional) is primitive recursively definable in **MBR + KBR**.

- Note that:
  - **FAN** is not S1-S9 computable in **C** and it is not majorizable.
  - **MBR** is not S1-S9 computable in **C** but it is majorizable.
  - **KBR** is S1-S9 computable in **C** but it is not majorizable.
Lemma[BO’02] Modified bar recursion can also be used to define (primitive recursively) the following search functional

\[ \tilde{\mu}(Y, \alpha^\rho, k) :\equiv \min n \geq k [Y(\overline{\alpha n} * 0) < n]. \]

Lemma[BO’02] \( SBR_{\rho, N} \) is primitive recursively definable in \( \tilde{\mu} + MBR_\rho \) (= MBR_\rho)

Lemma[BO’02] \( SBR_{\rho, \tau} \) is primitive recursively definable in \( SBR_{\rho', N} \), where if \( \tau = \tau_1 \rightarrow \ldots \rightarrow \tau_n \rightarrow N \) then \( \rho' = \rho \times \tau_1 \times \ldots \times \tau_n \).

Thm[BO’02] \( SBR \) is primitive recursively definable in \( MBR \).
Conclusions

Modified bar recursion