Modified bar recursion

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The Plan

1. Bar recursion:
   - Spector's bar recursion, (SBR)
   - Kohlenbach's bar recursion, (KBR)
   - Modified bar recursion. (MBR)
2. Interpreting classical Analysis ($\mathbf{PA}^\omega + \mathbf{AC}_0$) in $\mathbf{T}$ + MBR.
3. The model $\mathcal{M}$ and MBR.
4. Using MBR + KBR to define the fan functional.
   - S1-S9 computability.
5. Using MBR to define SBR.

Spector's bar recursion

- Spector's extension of Gödel's **T** consists of adding new functional symbols $\mathsf{SBR}_{\rho,\tau}$ to the language, and the axiom schema

$$\mathsf{SBR}_{\rho,\tau}(Y,G,H,s) \stackrel{\tau}{=} \begin{cases} G(s) & Y(s*\mathbf{0}) \stackrel{\mathbb{N}}{<} |s| \\ H(s,\lambda x^{\rho}.\mathsf{SBR}(s*x)) & \text{otherwise.} \end{cases}$$

  Thm[Spector'62] From a proof

  $$\mathbf{PA}^{\omega} + \mathbf{AC}_0 \vdash \forall x^{\sigma} \exists y^{\rho}\ A_{qf}(x,y),$$

  one can extract a closed term $t^{\sigma \to \rho}$ (of $\mathbf{HA}^{\omega} + \mathsf{SBR}$) such that

  $$\mathbf{HA}^{\omega} + \mathsf{SBR} \vdash A_{qf}(x, tx).$$

- Bezem'86 showed that $\mathcal{M}$ (the type structure of strongly majorizable functionals) is a model of SBR.

Kohlenbach's bar recursion

- In Kohlenbach'90 the following bar recursive functional is defined:

$$\mathsf{KBR}_{\rho,\tau}(Y, G, H, s) \stackrel{\tau}{=} \begin{cases} G(s) & Y(s * \mathbf{0}) \stackrel{\mathbb{N}}{=} Y(s * \mathbf{1}) \\ H(s, \lambda x^\rho.\mathsf{KBR}(s * x)) & \text{otherwise.} \end{cases}$$

- KBR defines SBR primitive recursively.
- The functional KBR is not majorizable, i.e. $\mathcal{M}$ is not a model of KBR. Hence, KBR is not primitive recursively definable in SBR.
- KBR is primitive recursively definable in SBR plus

$$\mu(Y, \alpha^{\mathbb{N}^\omega}, k) :\equiv \min n \geq k \; [Y(\overline{\alpha}n * \mathbf{0}) = Y(\overline{\alpha}n * \mathbf{1})].$$

Modified bar recursion

- Modified bar recursion is defined as

$$\text{MBR}(Y, H, s) \stackrel{\mathbb{N}}{=} Y(s * H(s, \lambda x^{\rho}.\text{MBR}(Y, H, s * x))).$$

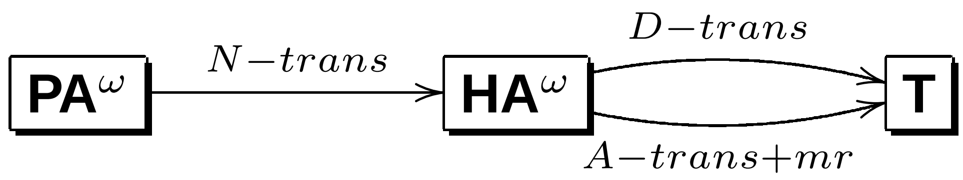
Thm[BO'02] From a proof

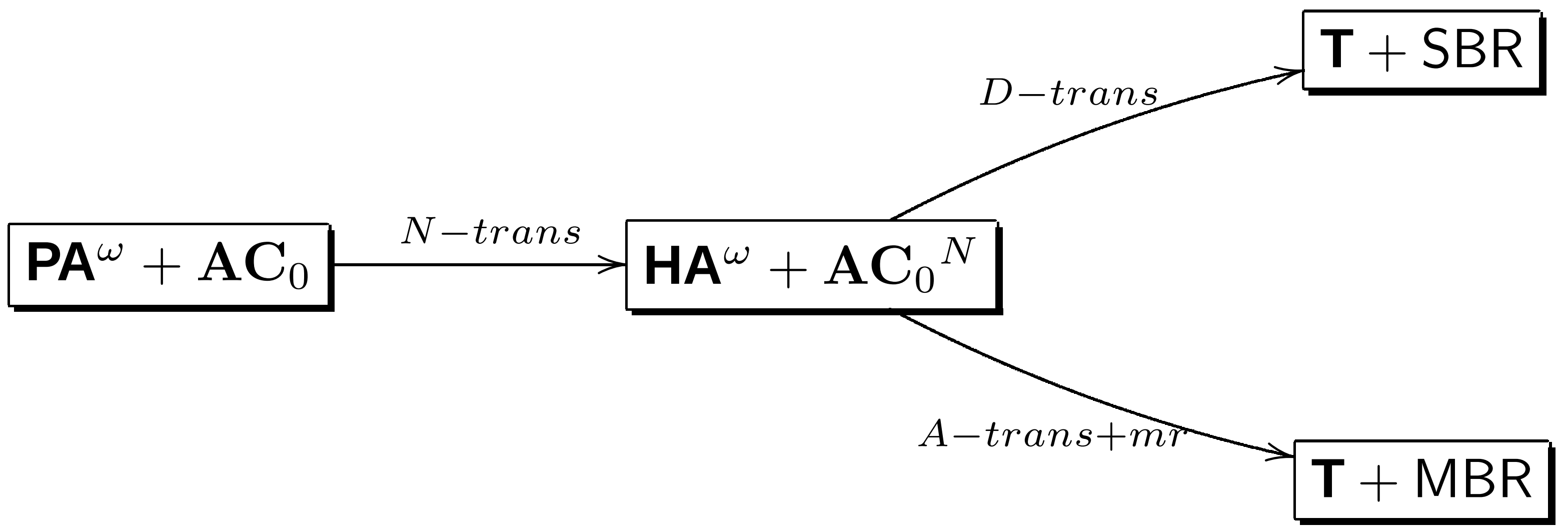
$$\text{PA}^{\omega} + \mathbf{AC}_0 \vdash \forall x^{\sigma} \, \exists y^{\mathbb{N}} \, A_{qf}(z, y)$$

where $A_{qf}(x, y)$ is quantifier free, one can extract a closed term $t^{\sigma \to \mathbb{N}}$ (of $\textbf{HA}^{\omega} + \text{MBR}$) such that

$$\textbf{HA}^{\omega} + \textbf{Continuity} + \text{MBR} \vdash A_{qf}(x, tx).$$

Interpreting Arithmetic and Analysis





The model of strongly majorizable functionals

Thm[BO'02] $\mathcal{M}$ is a model of modified bar recursion.

- We need two steps:

1. There exists a functional
   $$\Phi \in \mathcal{M}_{\rho^\omega \to \mathbb{N}} \times \mathcal{M}_{\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega} \times \mathcal{M}_{\rho^*} \to \mathcal{M}_{\mathbb{N}}$$
   satisfying the equation MBR.
2. There exists a functional Φ^* majorizing Φ, then
   $$\Phi \in \mathcal{M}_{(\rho^\omega \to \mathbb{N}) \times (\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega) \times \rho^* \to \mathbb{N}}$$

MBR in $\mathcal{M}$ (step 1.)

- We first prove a "weak continuity property" of the majorizable functionals of type $\rho^\omega \to \mathbb{N}$.

  Lemma[BO'02] The following is true in $\mathcal{M}$,

$$\forall Y^{\rho^\omega \to \mathbb{N}}, \alpha^{\rho^\omega} \exists n^{\mathbb{N}} \forall \beta^{\rho^\omega} \; (Y(\overline{\alpha}n * \beta) < n).$$

- Let $Y^{\rho^\omega \to \mathbb{N}}$ and $H^{\rho^* \times (\rho \to \mathbb{N}) \to \rho^\omega}$ (in $\mathcal{M}$) be fixed. We show that a Φ exists such that, for all s,

$$(*) \quad \Phi(s) = Y(s \mathbin{@} H(s, \lambda x.\Phi(s * x))).$$

- For a fixed α pick n as in the lemma above. Then for s extending $\overline{\alpha}n$ we must have $\Phi(s) \in \{0, \ldots, n\}$.

- By a fixed point argument we can show that a solution for equation $(*)$ exists for all s extending $\overline{\alpha}n$. By bar induction we get a solution for all s.

MBR in $\mathcal{M}$ (step 2.)

- A functional $\Omega : \rho \to \rho$ is defined in Kohlenbach'90 such that,
  i) For all F, $\Omega(F)$ maj F,
  ii) $\Omega \in \mathcal{M}$.
- By the weak continuity property of $\mathcal{M}$ we also get that the functional
  $$\Gamma(Y)(\alpha) :\equiv \min n \, [\forall \beta \in \overline{\alpha}n(\Omega(Y)(\beta) \leq n)]$$
  is well-defined. Moreover, $\Gamma \in \mathcal{M}$.
- We can then show that
  $$\Phi^* :\equiv \lambda Y, H.\Omega(\Phi \hat{Y} \hat{H}) \text{ maj } \Phi,$$
  majorizes Φ, where $\hat{Y}(\alpha) :\equiv \Gamma(Y)(\Omega(\alpha))$ and $\hat{H}(s, \alpha) :\equiv H(\Omega(s), \Omega(\alpha))$.

Fan functional

- A continuous functional $Y : \mathbb{N}^\omega \to \mathbb{N}$ on the Cantor subspace ($\{0,1\}^\omega$) is uniformly continuous.
- A fan functional $\Phi : (\{0,1\}^\omega \to \mathbb{N}) \to \mathbb{N}$ on input $Y : \{0,1\}^\omega \to \mathbb{N}$ should produce a point of uniform continuity n, i.e.

$$\forall \alpha, \beta \in \{0,1\}^\omega \ (\overline{\alpha}n = \overline{\beta}n \to Y(\alpha) = Y(\beta))$$

Thm[Tait] No fan functional is S1-S9 computable over $\mathcal{C}$ (the model of total continuous functionals).

[Berger] However, the minimal fan functional is S1-S9 computable over $\hat{\mathcal{C}}$ (the model of partial continuous functionals).

Defining the fan functional

- Using MBR one defines

$$\Phi(s,v) = s @ [\text{if } Y(\Phi(s*0,v)) \neq v \text{ then } \Phi(s*0,v) \\ \text{else } \Phi(s*1,v)]$$

- Using KBR (and the functional Φ) one defines

$$\Psi_Y(s) = \begin{cases} 0 & \text{if } Y(\hat{s}) \stackrel{\mathbb{N}}{=} Y(\Phi(s,v)) \\ 1 + \max\{\Psi_Y(s*0), \Psi_Y(s*1)\} & \text{otherwise,} \end{cases}$$

where $\hat{s} = s * \mathbf{0}$ and $v = Y(s * \mathbf{0})$.

- $\lambda Y.\Psi_Y(\langle\ \rangle)$ is a fan functional. Therefore, MBR is not S1-S9 computable in $\mathcal{C}$.

- Moreover, since Spector's bar recursion is S1-S9 computable in $\mathcal{C}$ we can conclude:

  Thm[BO'02] SBR does not define (primitive recursively) modified bar recursion.

Fan functional (cont.)

- FAN (the minimal fan functional) is primitive recursively definable in MBR $+$ KBR.
- Note that:
  - FAN is not S1-S9 computable in $\mathcal{C}$ and it is not majorizable.
  - MBR is not S1-S9 computable in $\mathcal{C}$ but it is majorizable.
  - KBR is S1-S9 computable in $\mathcal{C}$ but it is not majorizable.

Using MBR to define SBR

Lemma[BO'02] Modified bar recursion can also be used to define (primitive recursively) the following search functional

$$\tilde{\mu}(Y, \alpha^{\rho^\omega}, k) :\equiv \min n \geq k \, [Y(\overline{\alpha} n * \mathbf{0}) < n].$$

Lemma[BO'02] $\mathsf{SBR}_{\rho,\mathbb{N}}$ is primitive recursively definable in $\tilde{\mu} + \mathsf{MBR}_\rho$ ($= \mathsf{MBR}_\rho$)

Lemma[BO'02] $\mathsf{SBR}_{\rho,\tau}$ is primitive recursively definable in $\mathsf{SBR}_{\rho',\mathbb{N}}$, where if $\tau = \tau_1 \to \ldots \to \tau_n \to \mathbb{N}$ then $\rho' = \rho \times \tau_1 \times \ldots \times \tau_n$.

Thm[BO'02] SBR is primitive recursively definable in MBR.

Conclusions