# On the Feasibility of the CJ Three-User Interference Alignment Scheme for SISO OFDM Systems

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***Abstract*—The Cadambe-Jafar three-user interference alignment (CJ-3U-IA) scheme for single-input-single-output (SISO) systems is efficient to deal with wireless interference. The CJ-3U-IA scheme assumes that the channel fading coefficients of different time or frequency slots are independent. However, the independence of channel coefficients is not a necessary condition. For orthogonal frequency-division multiplexing (OFDM) systems, the channel coefficients of different frequency slots are generally not independent. In this letter, the necessary and sufficient conditions for the feasibility of the CJ-3U-IA scheme in three-user SISO OFDM systems are theoretically derived. An example has been shown that in the time domain, if a minimum of three channel impulse responses (CIRs) out of a total of nine CIRs have at least two significant taps while the other CIRs only have one significant tap, the CJ-3U-IA scheme is still feasible.**

***Index Terms*—Interference alignment, orthogonal frequency-division multiplexing (OFDM), single-input-single-output (SISO).**

## I. Introduction

THE Cadambe-Jafar three-user interference alignment (CJ-3U-IA) scheme for single-input-single-output (SISO) systems proposed in [1] is based on beamforming over multiple symbol extensions of time- or frequency-varying channels [2]-[3]. In the CJ-3U-IA scheme, the channel fading coefficients of different time or frequency slots are assumed to be independent. However, the independence of channel coefficients is not necessary [4]-[5]. For orthogonal frequency-division multiplexing (OFDM) systems, the frequency domain channel coefficients are the Fourier transform of time domain channel impulse responses (CIRs). Since the Fourier transform is linear, the frequency domain channel coefficients, which are the linear combination of time domain CIRs, are generally not independent. In a cellular network where OFDM is implemented, Suh et al show that for two-cell $K$-user interference multiple access channel scenario, when all the CIRs have at least two significant taps, each cell achieves the Degrees of Freedom $\frac{K}{K+1}$ almost surely [5].

To the best of our knowledge, the necessary and sufficient conditions for the CJ-3U-IA scheme for three-user SISO OFDM systems have not been derived to this date. In this letter, we theoretically derive the necessary and sufficient conditions for the feasibility of the CJ-3U-IA scheme for three-user SISO OFDM systems. We assume that the independence of the channel coefficients no longer holds. We employ the inverse Fourier transform to convert the problem from the frequency domain into the time domain and solve it.

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## II. System Model

Consider the three-user interference channel, comprised of three transmitters and three receivers. Each node is equipped with only one antenna. For each transmitter, the symbols transmitted over $2N + 1$ OFDM subcarriers are collectively denoted by a $(2N+1)\times 1$ signal vector $\overline{\mathbf{x}}_k$, where $k \in \{1, 2, 3\}$ is the user index. The frequency domain channel output at the $k^{th}$ receiver is described as follows

$$\mathbf{y}_k = \sum_{j=1}^{3} \mathbf{H}_{kj}\overline{\mathbf{x}}_j + \mathbf{z}_k, \tag{1}$$

where $\mathbf{y}_k$ is the $(2N + 1) \times 1$ output signal vector at the $k$th receiver, $\mathbf{H}_{kj}$, $j \in \{1, 2, 3\}$ is the $(2N+1)\times(2N+1)$ frequency domain channel fading matrix from Transmitter $j$ to Receiver $k$ and $\mathbf{z}_k$ is the $(2N + 1) \times 1$ additive white Gaussian noise (AWGN) vector at the $k^{th}$ receiver. The frequency domain channel fading matrix $\mathbf{H}_{kj}$ is a diagonal matrix whose diagonal entries are the Fourier transform of the time domain channel impulse responses (CIRs) $h_{kj}[n]$. $h_{kj}[n]$ is modeled by a tapped delay line model

$$h_{kj}[n] = \sum_{m=0}^{M_{kj}} \alpha_{kj}[m]\delta[n-m], \tag{2}$$

where $M_{kj}$ denotes the number of significant taps in $h_{kj}[n]$. We assume that all the terms $\left\{\alpha_{kj}[m]\right\}$ are independent zero-mean complex Gaussian with non-identical variances.

In the interference alignment scheme for the three-user SISO OFDM system, the transmitted signal vector, $\overline{\mathbf{x}}_1$, at Transmitter 1 is formed by the $(N+1)\times 1$ independent signal vector $\mathbf{x}_1$ multiplied by a $(2N + 1) \times (N + 1)$ precoder, $\mathbf{V}_1$, i.e., $\overline{\mathbf{x}}_1 = \mathbf{V}_1\mathbf{x}_1$. Similarly, the signal vectors $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_3$ at Transmitters 2 and 3, respectively, are formed by the $N \times 1$ independent signal vectors $\mathbf{x}_2$ and $\mathbf{x}_3$ multiplied by $(2N+1)\times N$ precoders, $\mathbf{V}_2$ and $\mathbf{V}_3$, i.e., $\overline{\mathbf{x}}_2 = \mathbf{V}_2\mathbf{x}_2$, and $\overline{\mathbf{x}}_3 = \mathbf{V}_3\mathbf{x}_3$. In [1], the precoders are designed as

$$\mathbf{V}_1 = \begin{bmatrix} \mathbf{w} & \mathbf{T}\mathbf{w} & \mathbf{T}^2\mathbf{w} & \cdots & \mathbf{T}^N\mathbf{w} \end{bmatrix}, \tag{3}$$

$$\mathbf{V}_3 = \mathbf{H}_{23}^{-1}\mathbf{H}_{21}\begin{bmatrix} \mathbf{T}\mathbf{w} & \mathbf{T}^2\mathbf{w} & \cdots & \mathbf{T}^N\mathbf{w} \end{bmatrix}, \text{ and} \tag{4}$$

$$\mathbf{V}_2 = \mathbf{H}_{32}^{-1}\mathbf{H}_{31}\begin{bmatrix} \mathbf{w} & \mathbf{T}\mathbf{w} & \cdots & \mathbf{T}^{N-1}\mathbf{w} \end{bmatrix}, \tag{5}$$

$$\begin{vmatrix} \hat{\kappa}_1\hat{\lambda}_1^N & \hat{\kappa}_1\hat{\lambda}_1^{N-1}\overline{\lambda}_1 & \cdots & \hat{\kappa}_1\overline{\lambda}_1^N & \overline{\kappa}_1\hat{\lambda}_1^N & \overline{\kappa}_1\hat{\lambda}_1^N\lambda_1 & \cdots & \overline{\kappa}_1\hat{\lambda}_1\lambda_1^{N-1} \\ \hat{\kappa}_2\hat{\lambda}_2^N & \hat{\kappa}_2\hat{\lambda}_2^{N-1}\overline{\lambda}_2 & \cdots & \hat{\kappa}_2\overline{\lambda}_2^N & \overline{\kappa}_2\hat{\lambda}_2^N & \overline{\kappa}_2\hat{\lambda}_2^N\lambda_2 & \cdots & \overline{\kappa}_2\hat{\lambda}_2\lambda_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\kappa}_{2N+1}\hat{\lambda}_{2N+1}^N & \hat{\kappa}_{2N+1}\hat{\lambda}_{2N+1}^{N-1}\overline{\lambda}_{2N+1} & \cdots & \hat{\kappa}_{2N+1}\overline{\lambda}_{2N+1}^N & \overline{\kappa}_{2N+1}\hat{\lambda}_{2N+1}^N & \overline{\kappa}_{2N+1}\hat{\lambda}_{2N+1}^N\lambda_{2N+1} & \cdots & \overline{\kappa}_{2N+1}\hat{\lambda}_{2N+1}\lambda_{2N+1}^{N-1} \end{vmatrix} = 0. \tag{16}$$

where

$$\mathbf{T} = \mathbf{H}_{12}\mathbf{H}_{21}^{-1}\mathbf{H}_{23}\mathbf{H}_{32}^{-1}\mathbf{H}_{31}\mathbf{H}_{13}^{-1} \text{ and} \tag{6}$$
$$\mathbf{w} = [\ 1 \quad 1 \quad \cdots \quad 1\ ]^\dagger \tag{7}$$

in which the superscript † denotes transpose.

## III. Feasibility Proof

In [1], the feasibility of the CJ-3U-IA scheme is proved by assuming that the diagonal entries of the diagonal matrix $\mathbf{H}_{kj}$ are independent. It is noted that for the proposed interference alignment scheme for the three-user SISO OFDM system, the diagonal entries in $\mathbf{H}_{kj}$ are not independent.

Consider the received signals at Receiver 1. The desired signals arrive along the $N+1$ vectors $\mathbf{H}_{11}\mathbf{V}_1$ while the interference arrives along the $N$ vectors $\mathbf{H}_{12}\mathbf{V}_2$ and the $N$ vectors $\mathbf{H}_{13}\mathbf{V}_3$. Since

$$\mathbf{H}_{12}\mathbf{V}_2 = \mathbf{H}_{13}\mathbf{V}_3, \tag{8}$$

it suffices to show that the $(2N+1)\times(2N+1)$ matrix

$$[\mathbf{H}_{11}\mathbf{V}_1\ \ \mathbf{H}_{12}\mathbf{V}_2] \tag{9}$$

is full-rank to prove that there are interference-free dimensions [1]. Multiplying (9) by the full-rank matrix $\mathbf{H}_{11}^{-1}$ and substituting the values of $\mathbf{V}_1$ and $\mathbf{V}_2$, we need to show that the matrix $\mathbf{S}$, which is defined as

$$\mathbf{S} \triangleq \left[\ \mathbf{w}\ \ \mathbf{Tw}\ \ \cdots\ \ \mathbf{T}^N\mathbf{w}\ \ \mathbf{Dw}\ \ \mathbf{DTw}\ \ \cdots\ \ \mathbf{DT}^{N-1}\mathbf{w}\ \right] \tag{10}$$

has almost surely full-rank where

$$\mathbf{D} = \mathbf{H}_{11}^{-1}\mathbf{H}_{12}\mathbf{H}_{32}^{-1}\mathbf{H}_{31} \tag{11}$$

is a diagonal matrix. In other words, we need to show that $|\mathbf{S}| \neq 0$ with probability 1. Define

$$[\lambda_1\ \lambda_2\ \cdots\ \lambda_{2N+1}]^\dagger = \mathbf{Tw}, \tag{12}$$
$$[\kappa_1\ \kappa_2\ \cdots\ \kappa_{2N+1}]^\dagger = \mathbf{Dw}. \tag{13}$$

We have

$$\mathbf{S} = \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^N & \kappa_1 & \kappa_1\lambda_1 & \cdots & \kappa_1\lambda_1^{N-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^N & \kappa_2 & \kappa_2\lambda_2 & \cdots & \kappa_2\lambda_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{2N+1} & \cdots & \lambda_{2N+1}^N & \kappa_{2N+1} & \kappa_{2N+1}\lambda_{2N+1} & \cdots & \kappa_{2N+1}\lambda_{2N+1}^{N-1} \end{bmatrix}. \tag{14}$$

Since the independence assumption of $\lambda_1, \lambda_2, \cdots, \lambda_{2N+1}$ and $\kappa_1, \kappa_2, \cdots, \kappa_{2N+1}$ of [1] no longer holds, we will employ the inverse Fourier transform to convert the problem from the frequency domain into the time domain and derive the necessary and sufficient conditions for the feasibility of the CJ-3U-IA scheme.

From (6) and (11), we observe that the matrices $\mathbf{T}$ and $\mathbf{D}$ contain both frequency domain channel fading matrix factors and inverse channel matrix factors. Thus, the effects of the channel matrix and the inverse channel matrix on the diagonal entries of $\mathbf{T}$ and $\mathbf{D}$, i.e., $\lambda_n$ and $\kappa_n$, $n \in \{1, 2, \cdots, 2N+1\}$, are coupled. To decouple the aforementioned effects, we decompose $\lambda_n$ and $\kappa_n$ as

$$\lambda_n = \frac{\overline{\lambda}_n}{\hat{\lambda}_n} \text{ and } \kappa_n = \frac{\overline{\kappa}_n}{\hat{\kappa}_n}, \tag{15}$$

where $\overline{\lambda}_n$ and $\overline{\kappa}_n$ correspond to the product of the frequency domain channel fading matrices that are not inverted; $\hat{\lambda}_n$ and $\hat{\kappa}_n$ correspond to the product of the matrices that are inverted. For example, from (11), $\overline{\kappa}_1, \overline{\kappa}_2, \cdots, \overline{\kappa}_{2N+1}$ are the diagonal entries of $\mathbf{H}_{12}\mathbf{H}_{31}$ and $\hat{\kappa}_1, \hat{\kappa}_2, \cdots, \hat{\kappa}_{2N+1}$ are the diagonal entries of $\mathbf{H}_{11}\mathbf{H}_{32}$. The decomposition is possible because the frequency domain channel fading matrices are diagonal and the multiplication of diagonal matrices does not depend on the multiplication sequence. Therefore, $|\mathbf{S}| = 0$ is equivalent to (16) shown on the top of this page. Let

$$\overline{\boldsymbol{\lambda}} = \left[\overline{\lambda}_1\ \overline{\lambda}_2\ \cdots\ \overline{\lambda}_{2N+1}\right]^\dagger, \tag{17}$$
$$\hat{\boldsymbol{\lambda}} = \left[\hat{\lambda}_1\ \hat{\lambda}_2\ \cdots\ \hat{\lambda}_{2N+1}\right]^\dagger, \tag{18}$$
$$\overline{\boldsymbol{\kappa}} = [\overline{\kappa}_1\ \overline{\kappa}_2\ \cdots\ \overline{\kappa}_{2N+1}]^\dagger, \text{ and} \tag{19}$$
$$\hat{\boldsymbol{\kappa}} = [\hat{\kappa}_1\ \hat{\kappa}_2\ \cdots\ \hat{\kappa}_{2N+1}]^\dagger. \tag{20}$$

The inverse Fourier transforms of $\overline{\boldsymbol{\lambda}}$, $\hat{\boldsymbol{\lambda}}$, $\overline{\boldsymbol{\kappa}}$, and $\hat{\boldsymbol{\kappa}}$ are denoted as $\overline{\boldsymbol{\xi}}$, $\hat{\boldsymbol{\xi}}$, $\overline{\boldsymbol{\zeta}}$, and $\hat{\boldsymbol{\zeta}}$, respectively. The column vectors $\overline{\boldsymbol{\xi}}$, $\hat{\boldsymbol{\xi}}$, $\overline{\boldsymbol{\zeta}}$, and $\hat{\boldsymbol{\zeta}}$ correspond to the time domain CIRs $h_{\overline{\xi}}[n]$, $h_{\hat{\xi}}[n]$, $h_{\overline{\zeta}}[n]$, and $h_{\hat{\zeta}}[n]$. Since $|\mathbf{S}| = 0$ is equivalent to

$$|\mathbf{FS}| = 0 \tag{21}$$

where $\mathbf{F}$ is the inverse Fourier transform operator, we get (22) shown on the top of next page. In (22), $\otimes$ denotes the circular convolution operation and

$$\mathbf{a}^{\otimes n} = \underbrace{\mathbf{a} \otimes \cdots \otimes \mathbf{a}}_{n}. \tag{23}$$

Since the channel is modeled as (2), the column vectors $\overline{\boldsymbol{\xi}}$, $\overline{\boldsymbol{\zeta}}$, $\hat{\boldsymbol{\xi}}$ and $\hat{\boldsymbol{\zeta}}$ can be divided into top and bottom parts where the top part includes all the nonzero elements and the bottom part includes all zero elements. Let $M_{\overline{\xi}}$, $M_{\hat{\xi}}$, $M_{\overline{\zeta}}$, and $M_{\hat{\zeta}}$ denote the number of nonzero elements in $\overline{\boldsymbol{\xi}}$ $\hat{\boldsymbol{\xi}}$, $\overline{\boldsymbol{\zeta}}$, and $\hat{\boldsymbol{\zeta}}$, respectively. We have the following lemma.

**Lemma 1** Consider two $(n+1)\times 1$ column vectors $\mathbf{a}$ and $\mathbf{b}$ that can be divided into top and bottom parts where the top part includes nonzero elements and the bottom part includes zero elements. The nonzero elements in $\mathbf{a}$ and $\mathbf{b}$ are random variables. Let $M_{\mathbf{a}}$ and $M_{\mathbf{b}}$ denote the number of nonzero elements in $\mathbf{a}$ and $\mathbf{b}$, respectively. The probability that the column vectors $\mathbf{a}$, $\mathbf{a}^{\otimes 2}$, $\cdots$, $\mathbf{a}^{\otimes n}$ and $\mathbf{b} \otimes \mathbf{a}^{\otimes k}$ where $0 \leq k \leq n$

$$\left| \hat{\zeta} \otimes \hat{\xi}^{\otimes N} \quad \hat{\zeta} \otimes \hat{\xi}^{\otimes N-1} \otimes \overline{\xi} \quad \cdots \quad \hat{\zeta} \otimes \overline{\xi}^{\otimes N} \quad \overline{\zeta} \otimes \hat{\xi}^{\otimes N} \quad \overline{\zeta} \otimes \hat{\xi}^{\otimes N-1} \otimes \overline{\xi} \quad \cdots \quad \overline{\zeta} \otimes \hat{\xi} \otimes \overline{\xi}^{\otimes N-1} \right| = 0 \tag{22}$$

$$C \cdot \left| \hat{\zeta}^* \otimes \mathbf{d}^{\otimes N+1} \quad \hat{\zeta}^* \otimes \mathbf{d}^{\otimes N} \quad \cdots \quad \hat{\zeta}^* \otimes \mathbf{d} \quad \overline{\zeta} \otimes \mathbf{d}^{\otimes N} \quad \overline{\zeta} \otimes \mathbf{d}^{\otimes N-1} \quad \cdots \quad \overline{\zeta} \otimes \mathbf{d} \right| = 0 \tag{26}$$

are linearly dependent is not equal to 0 if and only if:

1) the number of nonzero elements in $\mathbf{a}$ or $\mathbf{b}$ is less than 1, i.e., $M_{\mathbf{a}} \leq 1$ or $M_{\mathbf{b}} \leq 1$;
2) $M_{\mathbf{a}} = 2$ and $M_{\mathbf{b}} \leq n - k + 1$.

*Proof*: See Appendix A. ■

For OFDM systems, we analyze the feasibility of the CJ-3U-IA by considering whether or not $\overline{\xi}$ or $\hat{\xi}$ contains the same convolution factor as $\overline{\zeta}$ or $\hat{\zeta}$. In this letter, we say that $\mathbf{a}$ and $\mathbf{b}$ contain the same convolution factor if

$$\mathbf{a} = \mathbf{a}^* \otimes \mathbf{c} \text{ and } \mathbf{b} = \mathbf{b}^* \otimes \mathbf{c} \tag{24}$$

where $\mathbf{a}^*$, $\mathbf{b}^*$ and $\mathbf{c}$ are $(n + 1) \times 1$ column vectors whose nonzero elements are independent random variables.

It is noted that $\overline{\xi}$ does not contain the same convolution factor as $\hat{\xi}$. This is because $\overline{\xi}$ and $\hat{\xi}$ containing the same convolution factor would mean that $\mathbf{T}$ includes the factors $\mathbf{H}_{kj}$ and $\mathbf{H}_{kj}^{-1}$, which cancel each other out. Similarly, $\overline{\zeta}$ does not contain the same convolution factor as $\hat{\zeta}$.

When $\overline{\xi}$ or $\hat{\xi}$ does not contain the same convolution factor as $\overline{\zeta}$ or $\hat{\zeta}$, we have following theorem.

**Theorem 1** For OFDM systems, when $\overline{\xi}$ or $\hat{\xi}$ does not contain the same convolution factor as $\overline{\zeta}$ or $\hat{\zeta}$, the probability $\Pr(|\mathbf{S}| = 0) = 0$ if and only if:

1) the number of nonzero elements in $\overline{\xi}$ or $\hat{\xi}$ is larger than 1, i.e., $M_{\overline{\xi}} > 1$ or $M_{\hat{\xi}} > 1$;
2) $M_{\overline{\zeta}} > 1$ or $M_{\hat{\zeta}} > 1$;
3) when $M_{\hat{\xi}} = 1$ and $M_{\overline{\xi}} = 2$, the number of nonzero elements in $\overline{\zeta}$ is larger than $N + 1$, i.e., $M_{\overline{\zeta}} > N + 1$ or $M_{\hat{\zeta}} > N$;
4) when $M_{\overline{\xi}} = 1$ and $M_{\hat{\xi}} = 2$, the number of nonzero elements in $\overline{\zeta}$ or $\hat{\zeta}$ is larger than $N$;
5) when $M_{\overline{\xi}} + M_{\hat{\xi}} \geq 4$, it is required that $M_{\overline{\zeta}} > 1$ or $M_{\hat{\zeta}} > 1$.

*Proof*: If $M_{\overline{\xi}} = 1$ and $M_{\hat{\xi}} = 1$, $\hat{\zeta} \otimes \hat{\xi}^{\otimes n}$ and $\hat{\zeta} \otimes \overline{\xi}^{\otimes n}$ in (22) are linearly dependent. Thus, $|\mathbf{S}| = 0$. Similarly, if $M_{\overline{\zeta}} = 1$ and $M_{\hat{\zeta}} = 1$, we have $|\mathbf{S}| = 0$.

When $M_{\hat{\xi}} = 1$ and $M_{\overline{\xi}} = 2$, the column in matrix $\mathbf{FS}$ that contains the maximum number of nonzero elements may be $\hat{\zeta} \otimes \overline{\xi}^{\otimes N}$ or $\overline{\zeta}\hat{\xi}\overline{\xi}^{\otimes N-1}$. The number of nonzero elements in $\hat{\zeta} \otimes \overline{\xi}^{\otimes N}$ and $\overline{\zeta}\hat{\xi}\overline{\xi}^{\otimes N-1}$ is $M_{\hat{\zeta}} + N$ and $M_{\overline{\zeta}} + N - 1$, respectively. To ensure that $\mathbf{FS}$ has full rank, it is required that $M_{\overline{\zeta}} > N + 1$ or $M_{\hat{\zeta}} > N$. Furthermore, from Lemma 1, $M_{\overline{\zeta}} > N + 1$ or $M_{\hat{\zeta}} > N$ ensures that the column vectors $\hat{\zeta}$, $\hat{\zeta} \otimes \overline{\xi}$, $\cdots$, $\hat{\zeta} \otimes \overline{\xi}^{\otimes N}$ and $\overline{\zeta}$, $\overline{\zeta} \otimes \overline{\xi}$, $\cdots$, $\overline{\zeta} \otimes \overline{\xi}^{\otimes N-1}$ are linearly independent. Similarly, when $M_{\overline{\xi}} = 1$ and $M_{\hat{\xi}} = 2$, it is required that $M_{\overline{\zeta}} > N$ or $M_{\hat{\zeta}} > N$.

When $M_{\overline{\xi}} + M_{\hat{\xi}} \geq 4$, the number of nonzero elements in $\overline{\xi}^{\otimes N}$ or $\hat{\xi}^{\otimes N}$ is equal to $2N + 1$. Thus, to ensure that the matrix $\mathbf{FS}$ has full rank, the column pairs, such as $\hat{\zeta} \otimes \hat{\xi}^{\otimes N}$ and $\overline{\zeta} \otimes \hat{\xi}^{\otimes N}$, should not be linearly dependent. Since $\hat{\zeta}$ and $\overline{\zeta}$ do not contain the same convolution factor, to ensure that the matrix $\mathbf{FS}$ has full rank requires $\overline{\zeta}$ and $\hat{\zeta}$ not be linearly dependent, which means both $M_{\overline{\zeta}}$ and $M_{\hat{\zeta}}$ are not equal to 1, i.e., $M_{\overline{\zeta}} > 1$ or $M_{\hat{\zeta}} > 1$. ■

When $\overline{\xi}$ or $\hat{\xi}$ contains the same convolution factor as $\overline{\zeta}$ or $\hat{\zeta}$, by denoting $\overline{\xi}^*$, $\hat{\xi}^*$, $\overline{\zeta}^*$, $\hat{\zeta}^*$ as the vectors after removing the same convolution factor from $\overline{\xi}$, $\hat{\xi}$, $\overline{\zeta}$, $\hat{\zeta}$, respectively, we have following theorem where $\overline{\xi}$, $\hat{\xi}$, $\overline{\zeta}$, and $\hat{\zeta}$ are different from those in Theorem 1.

**Theorem 2** For OFDM systems, when $\overline{\xi}$ or $\hat{\xi}$ contains the same convolution factor as $\overline{\zeta}$ or $\hat{\zeta}$, the probability $\Pr(|\mathbf{S}| = 0) = 0$ if and only if:

1) the number of nonzero elements in $\overline{\xi}$ or $\hat{\xi}$ is larger than 1, i.e., $M_{\overline{\xi}} > 1$ or $M_{\hat{\xi}} > 1$;
2) $M_{\overline{\zeta}} > 1$ or $M_{\hat{\zeta}} > 1$;
3) when $M_{\hat{\xi}} = 1$ and $M_{\overline{\xi}} = 2$, if $\overline{\xi}$ contains the same convolution factor as $\overline{\zeta}$ or $\hat{\zeta}$, the number of nonzero elements in $\overline{\zeta}$ is larger than $N + 1$, i.e., $M_{\overline{\zeta}} > N + 1$ or $M_{\hat{\zeta}} > N$;
4) when $M_{\overline{\xi}} = 1$ and $M_{\hat{\xi}} = 2$, if $\hat{\xi}$ contains the same convolution factor as $\overline{\zeta}$, it is required that $M_{\hat{\zeta}} > N$ or $M_{\overline{\zeta}} > N$. If $\hat{\xi}$ contains the same convolution factor as $\hat{\zeta}$, it is required that $M_{\hat{\zeta}} > N + 1$ or $M_{\overline{\zeta}} > N + 1$;
5) when $M_{\overline{\xi}} + M_{\hat{\xi}} \geq 4$, it is required that $M_{\overline{\zeta}} > 1$ or $M_{\hat{\zeta}} > 1$ and when $M_{\overline{\zeta}}\left(M_{\hat{\zeta}}\right) > 1$, $\overline{\zeta}\left(\hat{\zeta}\right)$ is not linearly dependent with $\overline{\xi}$ and $\hat{\xi}$.

*Proof*: When $M_{\overline{\xi}} = 1$, $M_{\hat{\xi}} = 2$, and $\hat{\xi}$ contains the same convolution factor as $\hat{\zeta}$,

$$\hat{\xi} = \mathbf{d} \text{ and } \hat{\zeta} = \hat{\zeta}^* \otimes \mathbf{d}, \tag{25}$$

where $\mathbf{d}$ is a $(2N+1) \times 1$ column vector, (22) can be expressed as (26) shown on the top of this page, where $C$ is a scalar. The column in matrix $\mathbf{FS}$ that contains the maximum number of nonzero elements may be $\hat{\zeta}^* \otimes \mathbf{d}^{\otimes N+1}$ or $\overline{\zeta} \otimes \mathbf{d}^{\otimes N}$. The number of nonzero elements in $\hat{\zeta}^* \otimes \mathbf{d}^{\otimes N+1}$ and $\overline{\zeta} \otimes \mathbf{d}^{\otimes N}$ is $M_{\hat{\zeta}^*} + N + 1$ and $M_{\overline{\zeta}} + N$, respectively. To ensure that $\mathbf{FS}$ has full rank, it is required that $M_{\hat{\zeta}^*} > N - 1$ ($M_{\hat{\zeta}} > N$) or $M_{\overline{\zeta}} > N$. Furthermore, from Lemma 1, it is also required that $M_{\hat{\zeta}^*} > N - 1$ ($M_{\hat{\zeta}} > N$) or $M_{\overline{\zeta}} > N + 1$ to ensure that the column vectors $\hat{\zeta}^* \otimes \mathbf{d}^{\otimes N+1}$, $\hat{\zeta}^* \otimes \mathbf{d}^{\otimes N}$, $\cdots$, $\hat{\zeta}^* \otimes \mathbf{d}$ and $\overline{\zeta} \otimes \mathbf{d}^{\otimes N}$, $\overline{\zeta} \otimes \mathbf{d}^{\otimes N-1}$, $\cdots$, $\overline{\zeta} \otimes \mathbf{d}$ are linearly independent.

The other conditions can be verified by a similar method as in the proof of Theorem 1. ■

The above derivation considered the received signal vectors at Receiver 1. Similarly, considering the received signal vectors at Receivers 2 and 3, it is required that

$$[\mathbf{H}_{21}\mathbf{V}_1 \ \mathbf{H}_{22}\mathbf{V}_2] \text{ and } [\mathbf{H}_{31}\mathbf{V}_1 \ \mathbf{H}_{33}\mathbf{V}_3], \tag{27}$$

respectively, have full rank almost surely.

**Example**: The CJ-3U-IA scheme for three-user SISO OFDM systems is feasible when $h_{12}[n]$, $h_{23}[n]$ and $h_{31}[n]$ have at least two significant taps and other CIRs have one significant tap.

*Proof*: For Receiver 1, the time domain CIRs

$$h_{\overline{\xi}}[n] = h_{12}[n] \otimes h_{23}[n] \otimes h_{31}[n], \tag{28}$$
$$h_{\hat{\xi}}[n] = C_1, \tag{29}$$
$$h_{\overline{\zeta}}[n] = h_{12}[n] \otimes h_{31}[n], \text{ and} \tag{30}$$
$$h_{\hat{\zeta}}[n] = C_2, \tag{31}$$

where $h_{\overline{\xi}}[n]$, $h_{\hat{\xi}}[n]$, $h_{\overline{\zeta}}[n]$, and $h_{\hat{\zeta}}[n]$ correspond to the column vectors $\overline{\boldsymbol{\xi}}$, $\hat{\boldsymbol{\xi}}$, $\overline{\boldsymbol{\zeta}}$, and $\hat{\boldsymbol{\zeta}}$; $C_1$ and $C_2$ are

$$C_1 = h_{21}[n] \otimes h_{32}[n] \otimes h_{13}[n] \text{ and} \tag{32}$$
$$C_2 = h_{11}[n] \otimes h_{32}[n], \tag{33}$$

which are complex scalars. Thus, we have $M_{\overline{\xi}} \geq 4$, $M_{\hat{\xi}} = 1$, $M_{\overline{\zeta}} \geq 3$, and $M_{\hat{\zeta}} = 1$. It is noted that $\overline{\boldsymbol{\xi}}$ and $\overline{\boldsymbol{\zeta}}$ contain the same convolution factor. Using Theorem 2, we obtain that the probability of $|\mathbf{S}| = 0$ is zero. Similarly, we obtain that the probabilities of zero determinants of (27) are zero. Thus, the CJ-3U-IA scheme is feasible. ■

## IV. Conclusions

In this letter, we have theoretically derived the necessary and sufficient conditions for the feasibility of the CJ-3U-IA scheme for three-user SISO OFDM systems in the time domain. An example has been shown that in the time domain, if a minimum of three channel impulse responses (CIRs) out of a total of nine CIRs have at least two significant taps while the other CIRs only have one significant tap, the CJ-3U-IA scheme is still feasible.

## Appendix A
## Proof of Lemma 1

If $M_{\mathbf{a}} \leq 1$ or $M_{\mathbf{b}} \leq 1$, the column vectors $\mathbf{a}$, $\mathbf{a}^{\otimes 2}$, $\cdots$, $\mathbf{a}^{\otimes n}$ and $\mathbf{b} \otimes \mathbf{a}^{\otimes k}$ are linearly dependent. If $M_{\mathbf{a}} = 2$,

$$\mathbf{a} = [a_1 \ a_2 \ 0 \ \cdots \ 0]^{\dagger}. \tag{34}$$

Define

$$\mathbf{A}_1 = \left[\mathbf{a} \ \mathbf{a}^{\otimes 2} \ \cdots \ \mathbf{a}^{\otimes n}\right]. \tag{35}$$

The matrix $\mathbf{A}_1$ is an $(n+1) \times n$ matrix. Substituting (34) into (35), we have

$$\mathbf{A}_1 = \begin{bmatrix} a_1 & a_1^2 & a_1^3 & \cdots & a^n \\ a_2 & 2a_1a_2 & 3a_1^2a_2 & \cdots & na_1^{n-1}a_2 \\ 0 & a_2^2 & 3a_1a_2^2 & \cdots & na_1^{n-2}a_2^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_2^n \end{bmatrix}. \tag{36}$$

Subtract in order, $a_1$ times column 1 from column 2, $a_1$ times column 2 from column 3, and so on. Dividing columns $2, 3, \cdots, n$ by $a_2$, we obtain

$$\mathbf{A}_2 = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 0 & a_2 & 2a_1a_2 & \cdots & 2a_1^{n-2}a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_2^{n-1} \end{bmatrix}. \tag{37}$$

Subtract in order, $a_1$ times column 2 from column 3, $a_1$ times column 3 from column 4, and so on. Dividing columns $3, 4, \cdots, n$ by $a_2$, we obtain $\mathbf{A}_3$. Repeating the same process, we obtain $\mathbf{A}_n$ as follows

$$\mathbf{A}_n = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 0 & \cdots & 0 \\ 0 & a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_2 \end{bmatrix}, \tag{38}$$

whose columns are $n$ different vertical shifts of vector $[a_1 \ a_2]^{\dagger}$. If $M_{\mathbf{a}} = 2$, we know that the number of nonzero elements of $\mathbf{a}^{\otimes k}$ is $k+1$. Since we employ linear operations to transform $\mathbf{A}_1$ into $\mathbf{A}_n$, when $0 \leq k \leq n$, $\mathbf{a}^{\otimes k}$ is the combination of different vertical shifts of vector $[a_1 \ a_2]^{\dagger}$. If $M_{\mathbf{b}} \leq n - k + 1$, $\mathbf{b} \otimes \mathbf{a}^{\otimes k}$ is linear combination of $n$ different vertical shifts of vector $[a_1 \ a_2]^{\dagger}$ and is linearly dependent with $\mathbf{A}_1$.

If $M_{\mathbf{a}} = 3$, the $n+1$ linearly independent vectors

$$\mathbf{a}_1 = [a_1 \ a_2 \ a_3 \ 0 \ \cdots \ 0]^{\dagger}, \tag{39}$$
$$\mathbf{a}_2 = [0 \ a_1 \ a_2 \ a_3 \ 0 \ \cdots \ 0]^{\dagger}, \tag{40}$$
$$\cdots$$
$$\mathbf{a}_{n+1} = [a_2 \ a_3 \ 0 \ \cdots \ 0 \ a_1]^{\dagger} \tag{41}$$

form a basis. We have

$$\mathbf{b} \otimes \mathbf{a}^{\otimes k} = b_1 \cdot \mathbf{a}_1 + b_2 \cdot \mathbf{a}_2 + \cdots + b_{M_{\mathbf{b}}+k-1} \cdot \mathbf{a}_{M_{\mathbf{b}}+k-1}. \tag{42}$$

In (42), the probability that $b_i = 0$, $1 \leq i \leq M_{\mathbf{b}} + k - 1$, is zero. We also have $\mathbf{a} = \mathbf{a}_1$ and $\mathbf{a}^{\otimes 2} = a_1 \cdot \mathbf{a} + a_2 \cdot \mathbf{a}_2 + a_3 \cdot \mathbf{a}_3$. Since the number of nonzero elements in the column vectors $\mathbf{a}^{\otimes 3}$, $\mathbf{a}^{\otimes 4}$, $\cdots$, $\mathbf{a}^{\otimes n}$ is larger than 6 and $\mathbf{a}^{\otimes 3}$, $\mathbf{a}^{\otimes 4}$, $\cdots$, $\mathbf{a}^{\otimes n}$ are linearly independent, $\mathbf{a}_2$ and $\mathbf{a}_3$ cannot be obtained by the linear combination of $\mathbf{a}$, $\mathbf{a}^{\otimes 2}$, $\mathbf{a}^{\otimes 3}$, $\cdots$, $\mathbf{a}^{\otimes n}$. Thus, if $M_{\mathbf{b}} > 1$, the probability that the column vectors $\mathbf{a}$, $\mathbf{a}^{\otimes 2}$, $\cdots$, $\mathbf{a}^{\otimes n}$ and $\mathbf{b} \otimes \mathbf{a}^{\otimes k}$ are linearly dependent is equal to 0. If $M_{\mathbf{a}} > 3$ and $M_{\mathbf{b}} > 1$, we obtain the same result.

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