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Improved MUSIC Under the Coexistence of Both Circular and Noncircular Sources

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***Abstract*—In array signal processing, the inner structure of the signals could possibly be exploited to improve the performance of the direction-of-arrival (DOA) estimations. One typical scenario has been discussed in [P. Chage, Y. Wang, and J. Saillard, "A Root-MUSIC Algorithm for Non-Circular Sources," *Proc. Int. Conf. Acoustics, Speech, Signal Processing* (ICASSP), Salt Lake City, UT, May 2001, pp. 7–11], where all the incoming signals are supposed to be sent by noncircular PAM/BPSK sources. A modified MUSIC algorithm was then developed to improve the DOA estimation accuracy, as well as to increase the maximum number of detectable signals. However, it is more realistic that some users send noncircular symbols while other users still send circular symbols. In this correspondence, we develop an algorithm to cope with this more general scenario. By exploiting the redundancy existing in the noncircular signals, the proposed algorithm can still increase the maximum number of detectable signals and improve the performance accuracy compared to the conventional MUSIC algorithm. Simulation results clearly show the effectiveness of our proposed algorithm.**

***Index Terms*—Array signal processing, direction-of-arrival estimation, MUSIC, noncircular sources.**

I. INTRODUCTION

Direction-of-arrival (DOA) estimation of narrowband planewave signals has been intensively studied in the past few decades. Many high-resolution algorithms have been developed for this problem [2]–[5]. However, the structure of incoming signals has not been considered in these traditional algorithms.

Only recently have some works been proposed to take into account the available information about the incoming signals. For example, in [6], a method was derived for signals with known waveforms. Constant modulus (CM) signals have been studied in [7], where some algorithms were developed to the case of phase-modulated signals. Noncircular signals have been considered in [1] and [8], and modified MUSIC algorithms were developed by exploiting the complex conjugate counterpart of the received signals. Although the algorithms in [1] and [8] can increase the number of detectable directions as well as to improve the performance accuracy, it is more realistic that some users send noncircular signals while others still send circular signals. In this correspondence, we study this more general scenario and develop a new MUSIC-based DOA estimation algorithm. The proposed algorithm can improve the DOA estimation accuracy compared to the conventional MUSIC algorithm. Meanwhile, it also allows to increase the maximum number of detectable signals beyond the conventional MUSIC limit.

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II. Problem Formulation

A. Array Model

Assume that L narrowband plane wave signals from $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_L]$ are impinging on an array of M sensors. The data snapshots from M sensors can be described by the signal model

$$\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\theta})\tilde{\mathbf{s}}(t) + \mathbf{n}(t), \quad t = 1, 2, \ldots, N \tag{1}$$

where $\tilde{\mathbf{s}}(t)$ is the $L \times 1$ vector representing the signal waveforms at the reference sensor; $\mathbf{n}(t)$ is the $M \times 1$ vector of white circular complex Gaussian noise with zero-mean and variance σ^2; N is the number of available snapshots; matrix $\mathbf{A}(\boldsymbol{\theta})$, with the form

$$\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \ldots, \mathbf{a}(\theta_L)] \tag{2}$$

is the $M \times L$ directional matrix, and $\mathbf{a}(\theta_l), l = 1, \ldots, L$ is the steering vector corresponding to the lth signal. Let the first sensor be the reference sensor with coordinates (0,0) and assume the remaining sensors stay at positions (x_m, y_m), $m = 2, \ldots, M$. Then, $\mathbf{a}(\theta_l)$ can be expressed as

$$\mathbf{a}(\theta_l) = \left[1, e^{j\frac{2\pi}{\lambda}(x_2 \sin\theta_l + y_2 \cos\theta_l)}, \ldots, e^{j\frac{2\pi}{\lambda}(x_M \sin\theta_l + y_M \cos\theta_l)}\right]^T \tag{3}$$

where λ denotes the signal wave length. We assume that all the sources are in the far field. In this case, the signal vector $\tilde{\mathbf{s}}(t)$ can be further modeled as

$$\tilde{\mathbf{s}}(t) = \dot{\mathbf{B}}\dot{\mathbf{s}}(t) \tag{4}$$

where $\dot{\mathbf{B}}$ is a constant $L \times L$ diagonal matrix, whose lth diagonal element represents the effect of the channel on the lth signal (the channel is assumed to be fixed during the estimation period), and $\dot{\mathbf{s}}(t)$ is the $L \times 1$ vector representing the signals sent out from sources. Hence, the data model can be rewritten as

$$\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\theta})\dot{\mathbf{B}}\dot{\mathbf{s}}(t) + \mathbf{n}(t), \quad t = 1, \ldots, N. \tag{5}$$

Remark: The matrix $\mathbf{A}(\boldsymbol{\theta})$ is in general a full-column-rank matrix when $L \leq M$ [2] and the ambiguity study, where $\mathbf{A}(\boldsymbol{\theta})$ may not be full rank, is treated as an independent issue from the DOA estimation [9], [10]. In fact, most DOA estimation algorithms [2]–[5] are developed by assuming that the ambiguity arises with zero probability.[1]

B. Noncircular Signals

Circularity is an important property of random variables [11]. The concept of circularity directly comes from the geometrical interpretation of complex random variables. Here, we use only the first and the second orders statistical properties of the signals. For a complex random variable y, the only moments to be considered are the mean $\mathrm{E}\{y\}$, the covariance $\mathrm{E}\{yy^*\}$, and the elliptic covariance $\mathrm{E}\{yy\}$. A complex random variable is said to be circular at the order 2, if both the mean and the elliptic covariance equal zero. The second order statistical characteristics of y are so contained in its covariance $\mathrm{E}\{yy^*\}$. Circularity is a common hypothesis for narrowband signals analysis, but we can easily find numerous noncircular signals, like PAM or BPSK signals.

Similar to [1], we only consider the case where noncircular sources emit PAM/BPSK signals. Denote the number of the noncircular sources and the circular sources by L_R and L_C, respectively, with $L = L_R + L_C$. Without loss of generality, let the first L_R elements in $\dot{\mathbf{s}}(t)$ represent noncircular sources as $\dot{s}_{r_i}(t), i = 1, \ldots, L_R$, and the remaining L_C elements represent the circular sources as $\dot{s}_{c_j}(t), j = 1, \ldots, L_C$, respectively. Each $\dot{s}_{r_i}(t)$ can be further expressed as the multiplication of a constant complex number $\bar{b}_{r_i}$ and a real signal $s_{r_i}(t)$, where $s_{r_i}(t)$ is a valid symbol in the PAM/BPSK constellation and $\bar{b}_{r_i}$ represents a constant phase rotation; namely

$$\dot{s}_{r_i}(t) = \bar{b}_{r_i} s_{r_i}(t). \tag{6}$$

However, this property does not hold for the circular sources $\dot{s}_{c_j}(t)$. Define

$$\mathbf{s}(t) = \left[s_{r_1}(t), \ldots, s_{r_{L_R}}(t), \dot{s}_{c_1}(t), \ldots, \dot{s}_{c_{L_C}}(t)\right]^T. \tag{7}$$

The vector $\dot{\mathbf{s}}(t)$ can be rewritten as

$$\dot{\mathbf{s}}(t) = \bar{\mathbf{B}}\mathbf{s}(t) \tag{8}$$

where $\bar{\mathbf{B}}$ is a diagonal matrix

$$\bar{\mathbf{B}} = \mathrm{diag}\left\{\bar{b}_{r_1}, \ldots, \bar{b}_{r_{L_R}}, 1, \ldots, 1\right\}. \tag{9}$$

Substituting (8) into (4) and then into (1) yields

$$\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\theta})\mathbf{B}\mathbf{s}(t) + \mathbf{n}(t), \quad t = 1, \ldots, N \tag{10}$$

where $\mathbf{B} = \dot{\mathbf{B}}\bar{\mathbf{B}}$, and the first L_R entries in $\mathbf{s}(t)$ are all real.

III. Improved MUSIC Algorithm

A. Modified Array Model

Similar to [2], L_R and L_C are assumed to be known *a priori*. If $L_R = L$, the array model (10) is the one studied in [1]. Here, however, the vector $\mathbf{s}(t)$ contains both real and complex entries. For simplicity in notations, we omit $\boldsymbol{\theta}$ and t in the following study unless otherwise mentioned. The matrices $\mathbf{A}$ and $\mathbf{B}$ and the vector $\mathbf{s}$ in (10) can be rewritten as

$$\begin{aligned}\mathbf{A} &= \left[\mathbf{a}(\theta_{r_1}), \ldots, \mathbf{a}\left(\theta_{r_{L_R}}\right), \mathbf{a}(\theta_{c_1}) \ldots, \mathbf{a}\left(\theta_{c_{L_C}}\right)\right] \\ &= \left[\mathbf{a}_{r_1}, \ldots, \mathbf{a}_{r_{L_R}}, \mathbf{a}_{c_1}, \ldots, \mathbf{a}_{c_{L_C}}\right]\end{aligned} \tag{11}$$

$$\mathbf{B} = \mathrm{diag}\left\{b_{r_1}, \ldots, b_{r_{L_R}}, b_{c_1}, \ldots, b_{c_{L_C}}\right\} \tag{12}$$

$$\mathbf{s} = \left[s_{r_1}, \ldots, s_{r_{L_R}}, s_{c_1}, \ldots, s_{c_{L_C}}\right]^T \tag{13}$$

where $\mathbf{a}_{r_i}$ and $\mathbf{a}_{c_j}$ denote the steering vectors in $\mathbf{A}$ corresponding to the ith PAM/BPSK signal and the jth circular signal, respectively. For future use, we define new notations as

$$\mathbf{A}_1 = \left[\mathbf{a}_{r_1}, \ldots, \mathbf{a}_{r_{L_R}}\right] \tag{14}$$

$$\mathbf{A}_2 = \left[\mathbf{a}_{c_1}, \ldots, \mathbf{a}_{c_{L_C}}\right] \tag{15}$$

$$\mathbf{B}_1 = \mathrm{diag}\left\{b_{r_1}, \ldots, b_{r_{L_R}}\right\} \tag{16}$$

$$\mathbf{B}_2 = \mathrm{diag}\left\{b_{c_1}, \ldots, b_{c_{L_C}}\right\}. \tag{17}$$

We can combine the observed signal vector and its complex conjugate counterpart into a new vector

$$\breve{\mathbf{x}} = \begin{bmatrix}\mathbf{x} \\ \mathbf{x}^*\end{bmatrix} = \begin{bmatrix}\mathbf{A}\mathbf{B}\mathbf{s} \\ \mathbf{A}^*\mathbf{B}^*\mathbf{s}^*\end{bmatrix} + \begin{bmatrix}\mathbf{n} \\ \mathbf{n}^*\end{bmatrix} = \breve{\mathbf{A}}\breve{\mathbf{s}} + \breve{\mathbf{n}}, \tag{18}$$

[1]The ambiguity, especially the high-rank ambiguity, usually exists for arbitrary array shape. However, the probability of the ambiguity are normally considered as zero.

where

$$\breve{\mathbf{A}} = \left[\breve{\mathbf{a}}_{r_1}, \ldots, \breve{\mathbf{a}}_{r_{L_R}}, \breve{\mathbf{A}}_{c_1}, \ldots, \breve{\mathbf{A}}_{c_{L_C}}\right] \tag{19}$$

$$\breve{\mathbf{a}}_{ri} = \begin{bmatrix} b_{r_i}\mathbf{a}_{r_i} \\ b_{r_i}^*\mathbf{a}_{r_i}^* \end{bmatrix} \tag{20}$$

$$\breve{\mathbf{A}}_{c_j} = \begin{bmatrix} b_{c_j}\mathbf{a}_{c_j} & \mathbf{0} \\ \mathbf{0} & b_{cj}^*\mathbf{a}_{cj}^* \end{bmatrix} \tag{21}$$

$$\breve{\mathbf{s}} = \left[s_{r_1}, \ldots, s_{r_{L_R}}, s_{c_1}, s_{c_1}^*, \ldots, s_{c_{L_C}}, s_{c_{L_C}}^*\right]^T \tag{22}$$

$$\breve{\mathbf{n}} = [\mathbf{n}^T, \mathbf{n}^H]^T \tag{23}$$

and $\mathbf{0}$ represents the $M \times 1$ zero vector. For $\breve{\mathbf{A}}$ to be a tall matrix, $L_R + 2L_C < 2M$ is required. The ambiguity of $\breve{\mathbf{A}}$ can be analyzed by following the same procedure presented in [9], [10], which is beyond the scope of this correspondence. Therefore, similar to conventional DOA estimation works, we will directly make the following important assumption for the time being.

Assumption 1: When, $L_R + 2L_C \leq 2M$, matrix $\breve{\mathbf{A}}$ is of full column rank for any L_C, L_R, θ_{r_i}, θ_{c_j} with probability one.

B. Main Algorithm

The covariance matrix $\breve{\mathbf{R}}$ of the newly defined vector $\breve{\mathbf{x}}$ is given by

$$\breve{\mathbf{R}} \triangleq \mathrm{E}\{\breve{\mathbf{x}}\breve{\mathbf{x}}^H\} = \breve{\mathbf{A}}\mathrm{E}\{\breve{\mathbf{s}}\breve{\mathbf{s}}^H\}\breve{\mathbf{A}}^H + \sigma^2\mathbf{I} = \breve{\mathbf{A}}\breve{\mathbf{R}}_{\mathbf{s}}\breve{\mathbf{A}}^H + \sigma^2\mathbf{I} \tag{24}$$

where $\breve{\mathbf{R}}_{\mathbf{s}} = \mathrm{E}\{\breve{\mathbf{s}}\breve{\mathbf{s}}^H\}$ is the covariance matrix for $\breve{s}$. If sources are not fully correlated, $\breve{\mathbf{R}}_{\mathbf{s}}$ is a full rank matrix, and the eigendecomposition of $\breve{\mathbf{R}}$ can be written as

$$\breve{\mathbf{R}} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^H + \sigma^2\mathbf{G}\mathbf{G}^H \tag{25}$$

where the $(L_R + 2L_C) \times (L_R + 2L_C)$ diagonal matrix $\mathbf{\Lambda}$ contains the $(L_R + 2L_C)$ signal-subspace eigenvalues of $\breve{\mathbf{R}}$ and the columns of the $2M \times (L_R + 2L_C)$ matrix $\mathbf{U}$ contains the signal-subspace eigenvectors of $\breve{\mathbf{R}}$. In turn, the $2M \times (2M - L_R - 2L_C)$ matrix $\mathbf{G}$ contains the noise-subspace eigenvectors of $\breve{\mathbf{R}}$.

Since both $\breve{\mathbf{A}}$ and $\mathbf{U}$ span the signal subspace, which are orthogonal to the noise subspace spanned by the matrix $\mathbf{G}$, we derive the criteria for estimating the DOAs as follows.

DOAs for Noncircular Sources: For any direction from $\{\theta_{r_1}, \ldots, \theta_{r_{L_R}}\}$, the following equation holds:

$$\mathbf{G}^H\breve{\mathbf{a}}_{r_i} = \mathbf{G}^H\mathbf{V}(\theta_{r_i})\begin{bmatrix} b_{r_i} \\ b_{r_i}^* \end{bmatrix} = \mathbf{0} \tag{26}$$

where the $2M \times 2$ matrix $\mathbf{V}(\theta)$ is defined as

$$\mathbf{V}(\theta) \triangleq \begin{bmatrix} \mathbf{a}(\theta) & \mathbf{0} \\ \mathbf{0} & \mathbf{a}^*(\theta) \end{bmatrix}. \tag{27}$$

Let us define

$$\mathbf{Q}(\theta) = \mathbf{V}^H(\theta)\mathbf{G}\mathbf{G}^H\mathbf{V}(\theta). \tag{28}$$

Then $\mathbf{Q}(\theta)$ is rank deficient at θ_{r_i} (This does not mean that $\mathbf{Q}(\theta)$ is rank deficient only at θ_{r_i}). The following estimator can be used to estimate DOAs for noncircular sources:

$$f_{\mathrm{r}}(\theta) = \frac{1}{\det\{\mathbf{Q}(\theta)\}} = \frac{1}{\det\left\{\mathbf{V}^H(\theta)\mathbf{G}\mathbf{G}^H\mathbf{V}(\theta)\right\}}. \tag{29}$$

If searched over the region $\theta \in (-90^\circ, +90^\circ]$, the DOAs for noncircular sources can be obtained from peaks in $f_{\mathrm{r}}(\theta)$. It should be mentioned that the number of the columns of $\mathbf{G}$ should be no less than 2. Otherwise, $\mathbf{Q}(\theta)$ is rank deficient regardless of the value of θ. Therefore, the prerequisite to use (29) is $2M - L_R - 2L_C \geq 2$.

DOAs for Circular Sources: For any direction from $\{\theta_{c_1}, \ldots, \theta_{c_{L_C}}\}$, there are

$$\mathbf{G}^H\begin{bmatrix} b_{c_j}\mathbf{a}_{c_j} \\ \mathbf{0} \end{bmatrix} = \mathbf{0}, \quad \text{and} \quad \mathbf{G}^H\begin{bmatrix} \mathbf{0} \\ b_{c_j}^*\mathbf{a}_{c_j}^* \end{bmatrix} = \mathbf{0}. \tag{30}$$

Divide $\mathbf{G}$ as $\mathbf{G} = [\mathbf{G}_1^T, \mathbf{G}_2^T]^T$ where $\mathbf{G}_1$ and $\mathbf{G}_2$ are two matrices with equal dimensions. Equation (30) can be rewritten as

$$\mathbf{a}_{c_j}^H\mathbf{G}_1\mathbf{G}_1^H\mathbf{a}_{c_j} = 0 \tag{31}$$

and

$$\mathbf{a}_{c_j}^H\left(\mathbf{G}_2\mathbf{G}_2^H\right)^*\mathbf{a}_{c_j} = 0. \tag{32}$$

Lemma 1: Two (31) and (32) are equivalent.

Proof: The orthogonality of $\mathbf{G}$ and $\breve{\mathbf{A}}$ can be written as

$$\begin{aligned} &\left[\mathbf{G}_1^H, \mathbf{G}_2^H\right]\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{0}_{M\times L_C} \\ \mathbf{A}_1^* & \mathbf{0}_{M\times L_C} & \mathbf{A}_2^* \end{bmatrix} = \mathbf{0}, \\ \Rightarrow \quad &\left[\mathbf{G}_2^T, \mathbf{G}_1^T\right]\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{0}_{M\times L_C} \\ \mathbf{A}_1^* & \mathbf{0}_{M\times L_C} & \mathbf{A}_2^* \end{bmatrix} = \mathbf{0} \Rightarrow \tilde{\mathbf{G}}^H\breve{\mathbf{A}} = \mathbf{0}. \end{aligned} \tag{33}$$

Obviously, $\tilde{\mathbf{G}} = [\mathbf{G}_2^T, \mathbf{G}_1^T]^H$ is an orthonormal matrix. Since $\breve{\mathbf{A}}$ is a full-column-rank matrix, $\tilde{\mathbf{G}}$ will also span the noise subspace. From the uniqueness of the projection matrix onto a subspace, one can readily conclude that

$$\mathbf{P}_G = \mathbf{G}\mathbf{G}^H = \tilde{\mathbf{G}}\tilde{\mathbf{G}}^H. \tag{34}$$

Therefore, $\mathbf{G}_1\mathbf{G}_1^H = (\mathbf{G}_2\mathbf{G}_2^H)^*$ holds. ■

Hence, we can use the following estimator to estimate DOAs for circular sources:

$$f_{\mathrm{c}}(\theta) = \frac{1}{\mathbf{a}^H(\theta)\mathbf{G}_1\mathbf{G}_1^H\mathbf{a}(\theta)} \tag{35}$$

where θ is, again, searched over the region $\theta \in (-90^\circ, +90^\circ]$. Note that, the estimator (35) is quite similar to the one in conventional MUSIC except that $\mathbf{G}_1$, the half of the noise subspace matrix $\mathbf{G}$, is used in the estimator.

We find that the following results can be obtained from (30):

$$\begin{aligned} \mathbf{0} &= \mathbf{G}^H\begin{bmatrix} b_{c_j}\mathbf{a}_{c_j} & \mathbf{0} \\ \mathbf{0} & b_{c_j}^*\mathbf{a}_{c_j}^* \end{bmatrix} = \mathbf{G}^H\begin{bmatrix} \mathbf{a}_{c_j} & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_{c_j}^* \end{bmatrix}\begin{bmatrix} b_{c_j} & 0 \\ 0 & b_{c_j}^* \end{bmatrix} \\ &= \mathbf{G}^H\mathbf{V}\left(\theta_{c_j}\right)\begin{bmatrix} b_{c_j} & 0 \\ 0 & b_{c_j}^* \end{bmatrix}. \end{aligned} \tag{36}$$

Hence, the following equation holds for circular sources:

$$\det\left\{\mathbf{V}^H\left(\theta_{c_j}\right)\mathbf{G}\mathbf{G}^H\mathbf{V}\left(\theta_{c_j}\right)\right\} = 0. \tag{37}$$

Therefore, peaks will also appear in the estimator (29) at $\theta = \theta_{c_j}$. Note that this conclusion is made under the ideal case when the true covariance $\breve{\mathbf{R}}$ is obtained. Practically, we can only approximate $\breve{\mathbf{R}}$ by averaging over the finite samples. Therefore, the two estimators may not necessarily give the same results of θ_{c_j}. By noting that (29) can be deduced from (35) but not vice versa,[2] we prefer to use the estimator (35) to obtain DOAs of circular sources. Furthermore, the noncircular directions θ_{r_i} cannot be detected from (35) since (30) cannot be deduced from (26)–(29).

[2]Since the orthogonality between $\mathbf{G}$ and $\breve{\mathbf{A}}_{c_j}$ indicates the rank deficiency of $\mathbf{Q}(\theta_{c_j})$ but not vice versa, we may think (35) as a "stronger" or a better estimator to θ_{c_j} than (29).

Remarks:

- The uniqueness of DOA estimation from (29) and (35) need to be studied. We show in the Appendix that the following statements hold true with probability one:
  — $\{\theta_{r_i}, \theta_{c_j} | i = 1, \ldots, L_R, j = 1, \ldots, L_C\}$ are the only DOAs that make $\mathbf{Q}(\theta)$ drop rank;
  — only $\theta_{c_j}, j = 1, \ldots, L_C$ can be found from (35).
- The proposed estimators (29) and (35) can be used to discriminate noncircular sources from circular sources.
- As mentioned before, $L_R + 2L_C \leq 2M - 2$ must be satisfied to ensure that $\mathbf{Q}(\theta)$ does not trivially drop rank. Hence, our proposed method can estimate the directions of more than $M-1$ signals when the transmitted signals contain at least two noncircular PAM/BPSK signals. Furthermore, if L_C is zero, the proposed algorithm reduces to exactly the same algorithm in [1].
- When the uniform linear array (ULA) is used, namely $x_m = (m-1)d$ and $y_m = 0, m = 1, \ldots, M$, the highly efficient polynomial rooting method [12] can be applied for the proposed estimator (29) and (35).

IV. Simulation Results

In this section, we examine the performance of the proposed estimators under various scenarios. For all examples, a six-sensor ULA with interelement spacing $d = \lambda/2$ is employed.

A. Example 1

In the first example, we compare the performance of the proposed method and the traditional MUSIC algorithm under the coexistence of both circular and noncircular sources. The number of snapshots is taken as $N = 300$, and all the results are averaged over $N_k = 100$ Monte Carlo runs. The root mean-square error (RMSE) is defined as

$$\mathrm{RMSE} = \sqrt{\frac{1}{LN_k}\sum_{k=1}^{N_k} \|\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}\|^2} \tag{38}$$

where the subscript k refers to the kth simulation run.

We assume that four uncorrelated signals come from 0°, 10°, 30°, 50°, respectively, and consider three cases where there are one, two, and three noncircular sources, respectively. For the case with one noncircular source, the source coming from 0° is supposed to send BPSK symbols, while other sources send QPSK symbols; for the case with two noncircular sources, the sources from 0° and 10° send BPSK symbols; for the case with three noncircular sources, the sources from 0°, 10°, and 30° send BPSK symbols.

The performance RMSEs versus the signal-to-noise ratio (SNR) of the two algorithms are shown in Fig. 1. We see that the proposed method performs better than the traditional MUSIC algorithm whenever there exist noncircular sources. Moreover, it is noticed that the performance of the proposed method becomes better when the number of noncircular sources increases. This is a direct result from the increment in the dimension of the noise subspace. However, this phenomenon is not observed for traditional MUSIC method, whose performance is almost unaffected by changing the number of the noncircular sources.

We also investigate the performance of both algorithms by changing the number of the snapshots. It is expected that as the number of the snapshots increases, the performance of both algorithms becomes better. Fig. 2 shows the performance RMSEs versus the number of the snapshots. Clearly, the proposed algorithm outperforms the traditional MUSIC over all snapshot regions.

One interesting observation through the simulations is that the improvement in the estimation accuracy is achieved not only for DOAs

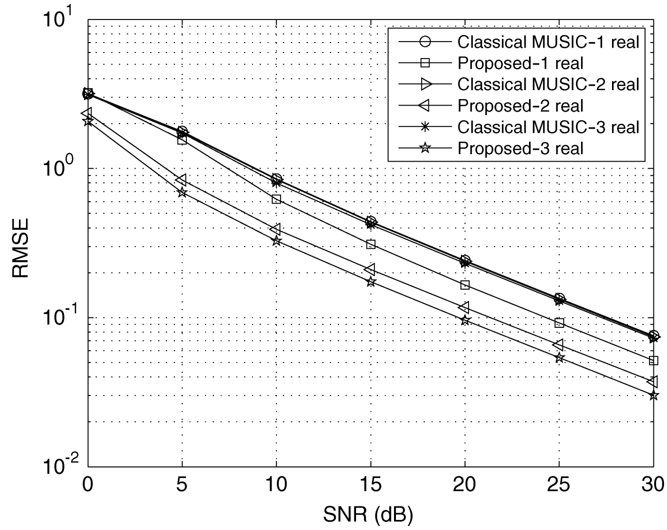

Fig. 1. RMSE versus SNR.

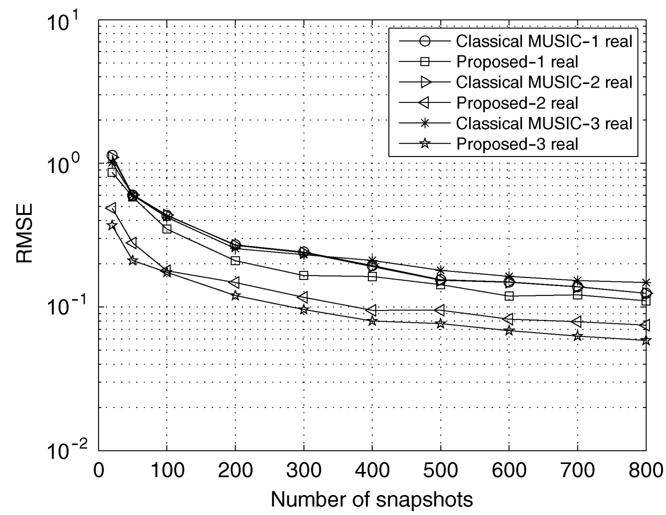

Fig. 2. RMSE versus number of snapshots.

of noncircular sources, but also for DOAs of circular sources. This is seen from Fig. 3.

B. Example 2

The second example studies the case where the number of incoming signals goes beyond the traditional MUSIC limit. We consider $L = 6$ uncorrelated signals coming from -40°, -20°, 0°, 10°, 30°, and 50°, respectively, whereas the symbols from the first four directions are obtained from BPSK modulation and others from QPSK modulation. The SNR of each source is taken as 20 dB. The dimension of the signal subspace can be computed as $L_R + 2L_C = 8$, which is smaller than the proposed limit $2M - 2 = 10$. Therefore, all six DOAs can be detected by the proposed method. The array patterns for both estimators (29) and (35) are shown in Fig. 4. One thing to be emphasized is that the traditional MUSIC algorithm fails to work under this case, since it cannot estimate more than $M - 1 = 5$ different directions.

Another interesting phenomenon is that the estimator (35) is not able to estimate the directions for noncircular sources, whereas the estimator (29) can detect both noncircular and circular sources. This is compatible with the theoretical analysis in Section III, where the complex estimator (35) is claimed only able to estimate DOAs for circular sources.

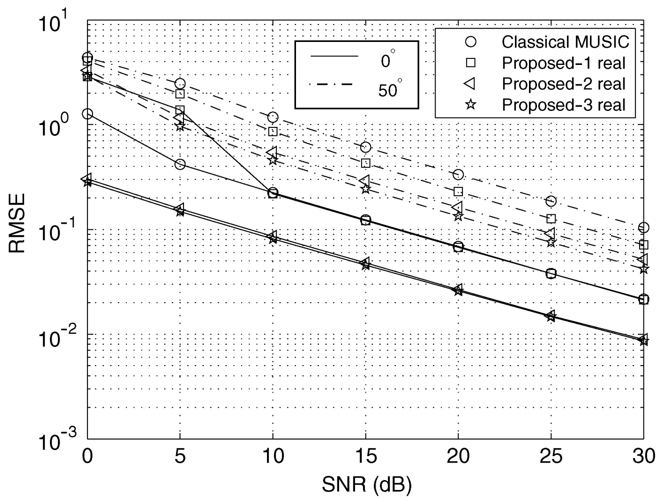

Fig. 3. Estimation of 0° and 50° under all three scenarios.

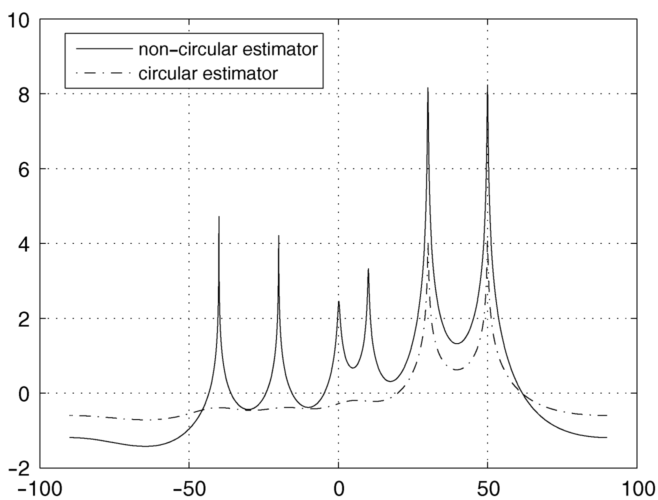

Fig. 4. Array pattern when the number of signals is beyond the traditional limit.

This property may be used to discriminate noncircular sources from circular sources under certain applications.

V. Conclusion

The problem of DOA estimation in the case of purely noncircular signals has been studied in [1], [8]. In this correspondence, we propose a MUSIC-based algorithm to cope with a more general scenario where both the circular and noncircular sources coexist. The proposed algorithm can achieve two important goals. First, it gives a more accurate estimate of DOAs in the situations where the number of sources is within the traditional limit of high resolution methods. Second, DOAs can still be estimated even when the number of sources is beyond the traditional limit. Finally, the computer simulations are provided to validate all the theoretical analysis clearly.

Appendix
Uniqueness of Proposed Estimators

A. Uniqueness of Estimator (29)

We first provide a prerequisite lemma.

Lemma 2: For arbitrary nonzero scalars g_1, g_2 with $|g_1| = |g_2|$, there exist another two scalars α, b, such that

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \alpha \begin{bmatrix} b \\ b^* \end{bmatrix}. \tag{39}$$

Proof: The scalars g_1, g_2, α, and b can be rewritten in the following forms:

$$g_1 = |g_1|e^{j\varepsilon_1}, \quad g_2 = |g_2|e^{j\varepsilon_2}, \quad \alpha = |\alpha|e^{j\varepsilon_\alpha}, \quad b = |b|e^{j\varepsilon_b}.$$

Scalars α, b can be calculated from (39) as

$$|\alpha||b| = |g_1| = |g_2| \tag{40}$$
$$\varepsilon_\alpha + \varepsilon_b = \varepsilon_1 \tag{41}$$
$$\varepsilon_\alpha - \varepsilon_b = \varepsilon_2. \tag{42}$$

Obviously, there always exist nonempty sets of α, b. ■

Lemma 3: If $\mathbf{Q}(\theta)$ drops rank at $\acute{\theta} \neq \theta_l, l = 1, \ldots, L$ (here we use θ_l instead of θ_{r_i} because all θ_l can make $\mathbf{Q}(\theta)$ rank deficient), then this $\acute{\theta}$ also satisfies (26); namely, there exists a scalar b, such that

$$\mathbf{G}^H \breve{\mathbf{a}}(\acute{\theta}) = \mathbf{G}^H \mathbf{V}(\acute{\theta}) \begin{bmatrix} b \\ b^* \end{bmatrix} = \mathbf{0}. \tag{43}$$

Proof: If $\mathbf{Q}(\acute{\theta})$ drops rank, then $\mathbf{G}^H\mathbf{V}(\acute{\theta})$, and consequently, $\mathbf{G}\mathbf{G}^H\mathbf{V}(\acute{\theta})$ are also column rank deficient. As a result, there exist scalars g_1 and g_2 with

$$\mathbf{G}\mathbf{G}^H\mathbf{V}(\acute{\theta}) \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \mathbf{0}. \tag{44}$$

From Lemma 1, we know

$$\mathbf{G}_1\mathbf{G}_1^H = \left(\mathbf{G}_2\mathbf{G}_2^H\right)^*, \quad \mathbf{G}_1\mathbf{G}_2^H = \left(\mathbf{G}_2\mathbf{G}_1^H\right)^*. \tag{45}$$

Substituting (45) into (44), we obtain

$$\mathbf{G}_1\mathbf{G}_1^H\mathbf{a}(\acute{\theta})g_1 + \mathbf{G}_1\mathbf{G}_2^H\mathbf{a}^*(\acute{\theta})g_2 = \mathbf{0} \tag{46}$$
$$\left(\mathbf{G}_1\mathbf{G}_2^H\right)^*\mathbf{a}(\acute{\theta})g_1 + \left(\mathbf{G}_1\mathbf{G}_1^H\right)^*\mathbf{a}^*(\acute{\theta})g_2 = \mathbf{0}. \tag{47}$$

The complex conjugate of (47) is written as

$$\mathbf{G}_1\mathbf{G}_1^H\mathbf{a}(\acute{\theta})g_2^* + \mathbf{G}_1\mathbf{G}_2^H\mathbf{a}^*(\acute{\theta})g_1^* = \mathbf{0}. \tag{48}$$

From (46) and (48), we know the following:

- if $[\mathbf{G}_1\mathbf{G}_1^H\mathbf{a}(\acute{\theta}), \mathbf{G}_1\mathbf{G}_2^H\mathbf{a}^*(\acute{\theta})]$ is a zero matrix, then there must exist a pair of g_1, g_2 with $|g_1| = |g_2|$;
- if $[\mathbf{G}_1\mathbf{G}_1^H\mathbf{a}(\acute{\theta}), \mathbf{G}_1\mathbf{G}_2^H\mathbf{a}^*(\acute{\theta})]$ is not a zero matrix, then, the dimension of its null space can only be one. Therefore

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \beta \begin{bmatrix} g_2^* \\ g_1^* \end{bmatrix}. \tag{49}$$

It can be easily obtained from (49) that $g_1 = |\beta|^2 g_1$, and then $|\beta| = 1$. Therefore, $|g_1| = |g_2|$ can be concluded.

From Lemma 2, we know that, there exists b, such that

$$\mathbf{G}\mathbf{G}^H\mathbf{V}(\acute{\theta}) \begin{bmatrix} b \\ b^* \end{bmatrix} = \mathbf{0}. \tag{50}$$

From (50)

$$\left([b,b^*]^H\mathbf{V}^H(\acute{\theta})\mathbf{G}\mathbf{G}^H\right)\left(\mathbf{G}\mathbf{G}^H\mathbf{V}(\acute{\theta})\begin{bmatrix} b \\ b^* \end{bmatrix}\right) = 0$$

$$\Rightarrow \quad [b,b^*]^H\mathbf{V}^H(\acute{\theta})\mathbf{G}\mathbf{G}^H\mathbf{V}(\acute{\theta})\begin{bmatrix} b \\ b^* \end{bmatrix} = 0 \qquad (51)$$

where the property that $\mathbf{P}_G = \mathbf{G}\mathbf{G}^H$ is the projection matrix is used. Note that (51) is in a quadrature form. We then conclude that

$$\mathbf{G}^H\mathbf{V}(\acute{\theta})\begin{bmatrix} b \\ b^* \end{bmatrix} = \mathbf{0}. \qquad (52)$$

■

Lemma 3 actually shows an equivalence between (26) and the estimator $f_r(\theta)$.

Let $\breve{\mathbf{a}}(\acute{\theta}) = [b\mathbf{a}(\acute{\theta}), b^*\mathbf{a}(\acute{\theta})^*]$. From Lemma 3 we know that if $\mathbf{Q}(\theta)$ drops rank at $\acute{\theta} \neq \theta_l$, then $\mathbf{G}^H\breve{\mathbf{a}}(\acute{\theta}) = \mathbf{0}$. Since $\breve{\mathbf{A}}$ is a full-column-rank matrix that is orthogonal to $\mathbf{G}$, the following statement can be equivalently made:

- there exists a $\acute{\theta} \neq \theta_l$, such that $\breve{\mathbf{a}}(\acute{\theta})$ is a linear combination of columns of $\breve{\mathbf{A}}(:,q)$, $q = 1,\ldots,L_R + 2L_C$.

From Assumption 1, we know the probability for the existence of $\acute{\theta}$ can be reasonably assumed to be zero.

Uniqueness of Estimator (35)

If there exists a $\grave{\theta} \neq \theta_{c_j}$ and $\mathbf{a}^H(\grave{\theta})\mathbf{G}_1\mathbf{G}_1^H\mathbf{a}(\grave{\theta}) = 0$. Then the following equations hold:

$$\mathbf{G}^H\begin{bmatrix} \mathbf{a}(\grave{\theta}) \\ \mathbf{0} \end{bmatrix} = \mathbf{0}, \quad \mathbf{G}^H\begin{bmatrix} \mathbf{0} \\ \mathbf{a}(\grave{\theta})^* \end{bmatrix} = \mathbf{0}. \qquad (53)$$

An equivalent statement is made as follows:

- there exists a $\grave{\theta} \neq \theta_{c_j}$, such that $[\mathbf{a}^T(\grave{\theta}), \mathbf{0}^T]^T$, and $[\mathbf{0}^T, \mathbf{a}^H(\grave{\theta})]^T$ is a linear combination of columns of $\breve{\mathbf{A}}(:,q)$, $q = 1,\ldots,L_R + 2L_C$.

Similarly, the probability for the existence of $\grave{\theta}$ is considered as zero, and the uniqueness for (35) can be guaranteed with probability one.

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Stochastic Maximum-Likelihood DOA Estimation in the Presence of Unknown Nonuniform Noise

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***Abstract*—This correspondence investigates the direction-of-arrival (DOA) estimation of multiple narrowband sources in the presence of nonuniform white noise with an arbitrary diagonal covariance matrix. While both the deterministic and stochastic Cramér-Rao bound (CRB) and the deterministic maximum-likelihood (ML) DOA estimator under this model have been derived by Pesavento and Gershman, the stochastic ML DOA estimator under the same setting is still not available in the literature. In this correspondence, a new stochastic ML DOA estimator is derived. Its implementation is based on an iterative procedure which concentrates the log-likelihood function with respect to the signal and noise nuisance parameters in a stepwise fashion. A modified inverse iteration algorithm is also presented for the estimation of the noise parameters. Simulation results have shown that the proposed algorithm is able to provide significant performance improvement over the conventional uniform ML estimator in nonuniform noise environments and require only a few iterations to converge to the nonuniform stochastic CRB.**

***Index Terms*—Direction-of-arrival (DOA) estimation , nonuniform noise, sensor array processing, stochastic maximum likelihood (ML) algorithm.**

I. Introduction

Direction–of-arrival (DOA) estimation has been one of the central problems in radar, sonar, navigation, geophysics, and acoustic tracking. A wide variety of high-resolution narrowband DOA estimators have been proposed and analyzed in the past few decades [2]–[5]. The maximum likelihood (ML) estimator, which shows excellent asymptotic performance, plays an important role among these techniques. Many of the proposed ML estimators are derived from the *uniform white noise assumption* [5]–[7], in which the noise process of each sensor is assumed to be spatially uncorrelated white Gaussian with identical unknown variance. It is shown that under this assumption the estimates of the nuisance parameters (source waveforms and noise variance) can be expressed as a function of DOAs [8]–[10], and, therefore, the number of independent parameters to be estimated is substantially reduced. This

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