# Fuzzy Impulsive Control of High-Order Interpolative Low-Pass Sigma–Delta Modulators

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***Abstract*—In this paper, a fuzzy impulsive control strategy is proposed. The state vectors that the impulsive controller resets to are determined so that the state vectors of interpolative low-pass sigma–delta modulators (SDMs) are bounded within any arbitrary nonempty region no matter what the input step size, the initial condition and the filter parameters are, the occurrence of limit cycle behaviors and the effect of audio clicks are minimized, as well as the state vectors are close to the invariant set if it exists. To work on this problem, first, the local stability criterion and the condition for the occurrence of limit cycle behaviors are derived. Second, based on the derived conditions, as well as a practical consideration based on the boundedness of the state variables and a heuristic measure on the strength of audio clicks, fuzzy membership functions and a fuzzy impulsive control law are formulated. The controlled state vectors are then determined by solving the fuzzy impulsive control law. One of the advantages of the fuzzy impulsive control strategy over the existing linear control strategies is the robustness to the input signal, the initial condition and the filter parameters, and that over the existing nonlinear control strategy are the efficiency and the effectiveness in terms of lower frequency of applying the control force and higher signal-to-noise ratio (SNR) performance.**

***Index Terms*—Fuzzy impulsive control, high order, interpolative sigma–delta modulators (SDMs).**

## I. Introduction

THE sigma–delta modulation technique has been proposed and applied in analog-to-digital (A/D) and digital-to-analog (D/A) conversion for many years [1]. It is particularly popular in the past few years because of the advance in electronic technology that makes the devices practical with low implementation cost [2]. Since sigma–delta modulators (SDMs) can achieve very high signal-to-noise ratios (SNRs), it is widely applied in many systems required A/D and D/A conversions, such as in the consumer and professional audio processing systems [2], communication systems [3], and precision measurement devices [4].

In order to improve the SNR, high-order SDMs are preferred. However, high-order SDMs suffer from instability problems. Although there are many existing linear control strategies for stabilizing interpolative SDMs, such as variable structure compensation (sliding mode control strategy) [5] and time delay feedback control strategy [6], etc, these linear control strategies stabilize the loop filter by changing the effective poles of the loop filter. Since the loop filter is usually designed to have a very high SNR, it is not guaranteed that the SNR of the controlled SDMs is still maintained or even improved if the effective poles of the loop filter are changed. Moreover, the parameters in the controller depend on the loop filter parameters, so it is not guaranteed that a particular class of controllers can stabilize all types of interpolative SDMs. Furthermore, the controlled SDMs may still be unstable when the magnitude of the input signal is increased. In addition, it cannot be guaranteed that the controlled SDMs are stable for all initial conditions in the state space.

In order to control the SDMs without changing the effective poles of the loop filter, nonlinear control strategy, such as the clipping control strategy, was employed [2]. For the clipping control strategy, as the state variables are always reset to the *same* values, periodic output sequences may result and this periodic behavior is known as limit cycle behavior. This situation is found very frequently when the input signal is very slow time varying or the clipped level is set at very low value. For audio applications [2], the occurrence of limit cycle behaviors results to the annoying audio tones, which should be avoided. Besides, there may be a large jump between the unclipped and clipped state levels. As a result, audio clicks may be observed, which should also be avoided. Furthermore, as the set of the state vectors under the clipping control strategy is usually not the same as the invariant set, the clipping force may be applied very frequently.

In order to solve these problems, an impulsive control strategy is proposed in this paper, in which it is to reset the state vectors to *different* positions in the state space whenever the control force is applied. Hence, the occurrence of limit cycle behaviors and the effect of audio clicks can be minimized with the guarantee of the bounded state variables. Moreover, if the invariant set exists, then we only need to reset the state variables of the loop filter once and the state vectors of the SDMs are guaranteed to be within the invariant set forever if the effects of limit cycle behaviors and audio clicks do not consider. However, there are usually an infinite number of state vectors in the invariant set, this paper is to determine the state vectors that the impulsive controller resets to. Since the SDMs consist of a quantizer, nonlinear behaviors, such as fractal and chaotic behaviors, combined with the practical consideration on the boundedness of the state variables and a heuristic measure on the strength of audio clicks, cause a difficulty to solve the state vectors analytically. To solve this problem, a fuzzy approach is employed because employing fuzzy approach can simplify the complicated problems and capture heuristic knowledge in the system.

The outline of this paper is as follows. In Section II, we introduce the notations which appear throughout this paper. In

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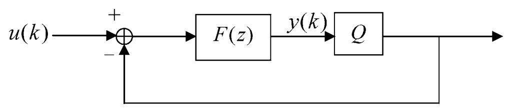


Fig. 1. Block diagram of an interpolative SDM.

Section III, the conditions for the occurrence of limit cycle behaviors and the local stability criterion of the SDMs are derived, which are used for the formulation of fuzzy membership functions and fuzzy impulsive control law. In Section IV, a fuzzy impulsive control strategy is proposed. In Section V, some simulation results are presented to illustrate the effectiveness of the fuzzy impulsive control strategy. Finally, a conclusion is summarized in Section VI.

## II. Notations

The block diagram of an interpolative SDM is shown in Fig. 1. The input to the SDM and the output of the loop filter are denoted as, respectively, $u(k)$ and $y(k)$. We assume that the loop filter is a single input single output real system and the input is also real, that is, $u(k) \in \Re$, so $y(k) \in \Re$. The transfer function of the loop filter is denoted as $F(z)$. $F(z)$ is assumed to be causal, rational and proper with the order of the polynomial of $z^{-1}$ in the numerator being equal to that in the denominator and there is a delay in the numerator. We make those assumptions because this type of SDMs is commonly used in the industry [2]. Denote the coefficients in the denominator and numerator of $F(z)$ as, respectively, $a_i$ for $i = 0, 1, \ldots, N$ and $b_j$ for $j = 1, \ldots, N$, where $N$ is the order of the loop filter. Then

$$F(z) = \frac{\sum_{j=1}^{N} b_j z^{-j}}{\sum_{i=0}^{N} a_i z^{-i}}. \tag{1}$$

Since this paper is based on the feedforward structure of the SDMs, without loss of generality, we assume that the loop filter is realized via the direct form because the expressions will be much simplified. For other minimal realizations, they can be converted to the direct form realization using simple transformations. Hence, the SDMs can be described by the following state space equation:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\left(\mathbf{u}(k) - \mathbf{s}(k)\right) \tag{2}$$

for $k \geq 0$, where

$$\mathbf{x}(k) \equiv [x_1(k), \quad \ldots, \quad x_N(k)]^T \equiv [y(k-N), \quad \ldots, \quad y(k-1)]^T \tag{3}$$

is the state vector of the SDMs

$$\mathbf{u}(k) \equiv [u(k-N), \quad \ldots, \quad u(k-1)]^T \tag{4}$$

$$\mathbf{s}(k) \equiv [s_1(k), \quad \ldots, \quad s_N(k)]^T \equiv [Q(y(k-N)), \quad \ldots, \quad Q(y(k-1))]^T \tag{5}$$

$$\mathbf{A} \equiv \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\frac{a_N}{a_0} & \cdots & \cdots & \cdots & -\frac{a_1}{a_0} \end{bmatrix}$$
$$\mathbf{B} \equiv \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \\ \frac{b_N}{a_0} & \cdots & \cdots & \cdots & \frac{b_1}{a_0} \end{bmatrix} \tag{6}$$

in which $Q$ is a one bit quantizer defined as follows:

$$Q(y) \equiv \begin{cases} 1, & y \geq 0 \\ -1, & \text{otherwise.} \end{cases} \tag{7}$$

Since the oversampling ratio of the SDM is usually very high, the input can be approximated as a step signal. Hence, we further assume that $\mathbf{u}(k) = \bar{\mathbf{u}}$ for $k \geq 0$.

In many practical situations, the magnitude of the state variables of the SDM should not be larger than certain values. For the direct form realization, since all the state variables are the delay versions of the output of the loop filter, we denote the desired bound on the state variables as $V_{\text{cc}}$. That is, $|x_i(k)| < V_{\text{cc}}$ for $i = 1, 2, \ldots, N$ and $k \geq 0$. Otherwise, the SDM is guaranteed to yield an unwanted behavior. Denote $B_o$ as the set of the desired state vectors. That is, $B_o \equiv \{\mathbf{x} : |x_i| < V_{\text{cc}} \text{ for } i = 1, 2, \ldots, N\}$.

## III. Conditions for Occurrence of Limit Cycle Behaviors and Local Stability Criterion

As discussed in Section I, limit cycle behaviors should be avoided. Hence, before we propose the fuzzy impulsive control strategy, the conditions for exhibiting limit cycle behavior are discussed below. This is essential for formulating a fuzzy membership function for avoiding the occurrence of limit cycle behavior.

Suppose the eigen decomposition of matrix $\mathbf{A}$ exists. That is, there exists a full rank matrix $\mathbf{T}$ and a diagonal matrix $\mathbf{D}$ which consist of the eigenvectors and eigenvalues of matrix $\mathbf{A}$, respectively, such that $\mathbf{A} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1}$. We make this assumption because it is satisfied for most of SDMs employed in the industry [2]. Denote $\lambda_i$ and $\boldsymbol{\xi}_i$ for $i = 1, 2, \ldots, N$ be the eigenvalues and the corresponding eigenvectors of the matrix $\mathbf{A}$. Let $n_d$ be the number of eigenvalues of matrix $\mathbf{A}$ on the unit circle with their phases are integer multiples of $(2\pi)/(P)$, that is, $\lambda_{i+N-n_d} = e^{(j2\pi k_i)/(P)}$ for $k_i \in Z$ and $i = 1, 2, \ldots, n_d$. Denote $L_i$ for $i = 1, 2, \ldots, N$ be the $i$th row of

$$\sum_{j=0}^{P-1} \mathbf{A}^{P-1-j}\mathbf{B}(\mathbf{u}(k_0+j) - \mathbf{s}(k_0+j)) \tag{8}$$

where $P \in Z^+$ and $k_0 \geq 0$. Let $\mathbf{r}_j$ for $j = 1, 2, \ldots, N$ be the $j$th row of $\mathbf{I} - \mathbf{A}^P$, where $\mathbf{I}$ is an $N \times N$ identity matrix. Denote

$$\Psi_P \equiv \{\, \mathbf{x}(0) : \mathbf{r}_i\mathbf{x}(k_0) = L_i, \qquad \text{for } i = 1, 2, \ldots, N - n_d\}. \tag{9}$$

*Lemma 1:* The number of linearly independent rows in the matrix $\mathbf{I}-\mathbf{A}^P$ is $N-n_d$, that is, $\exists c_{i,n} \in \Re$ for $i=1,2,\ldots,N-n_d$ and $n=1,2,\ldots,n_d$ such that $\sum_{i=1}^{N-n_d} c_{i,n}\mathbf{r}_i = \mathbf{r}_{N-n_d+n}$. If $\Psi_P \neq \oslash$, where $\oslash$ denotes the empty set, and $\sum_{i=1}^{N-n_d} c_{i,n}L_i = L_{N-n_d+n}$ for $n=1,2,\ldots,n_d$, then the SDMs exhibit limit cycle behavior with period $P$, and $\Psi_P$ is the corresponding nonempty set of initial condition. If $\Psi_P=\oslash$ or $\exists n \in \{1,2,\ldots,n_d\}$ such that $\sum_{i=1}^{N-n_d} c_{i,n}L_i \neq L_{N-n_d+n}$, then there will not exist any fixed point or periodic state sequence.

*Proof:* Denote $\mathbf{Q} \equiv \mathbf{I}-\mathbf{A}^P$. Since $\mathbf{A}=\mathbf{TDT}^{-1}$ and $\lambda_{i+N-n_d}=e^{(j2\pi k_i)/(P)}$ for $k_i \in Z$ and $i=1,2,\ldots,n_d$, we have

$$\mathbf{QT}=\left[\left(1-\lambda_1^P\right)\boldsymbol{\xi}_1,\ldots,\left(1-\lambda_{N-n_d}^P\right)\boldsymbol{\xi}_{N-n_d},\mathbf{0},\ldots,\mathbf{0}\right] \tag{10}$$

and

$$\operatorname{rank}(\mathbf{QT}) = \operatorname{rank}\left(\left[\left(1-\lambda_1^P\right)\boldsymbol{\xi}_1,\ldots,\left(1-\lambda_{N-n_d}^P\right)\boldsymbol{\xi}_{N-n_d},\mathbf{0},\ldots,\mathbf{0}\right]\right). \tag{11}$$

Since $\mathbf{T}$ is a full rank matrix, $\{\boldsymbol{\xi}_1,\ldots,\boldsymbol{\xi}_{N-n_d}\}$ are linearly independent. As $1-\lambda_i^P \neq 0$ for $i=1,2,\ldots,N-n_d$, $\operatorname{rank}(\mathbf{QT})=N-n_d$. However, $\operatorname{rank}(\mathbf{QT}) \le \operatorname{rank}(\mathbf{Q})$. Hence, $\operatorname{rank}(\mathbf{Q}) \ge N-n_d$. Since

$$\mathbf{Q}=\left[\left(1-\lambda_1^P\right)\boldsymbol{\xi}_1,\ldots,\left(1-\lambda_{N-n_d}^P\right)\boldsymbol{\xi}_{N-n_d},\mathbf{0},\ldots,\mathbf{0}\right]\mathbf{T}^{-1} \tag{12}$$

$\operatorname{rank}(\mathbf{Q}) \le N-n_d$. Hence, $\operatorname{rank}(\mathbf{Q})=N-n_d$. As a result, the number of linearly independent rows in the matrix $\mathbf{I}-\mathbf{A}^P$ is $N-n_d$.

Since $\Psi_P \neq \oslash$, $\exists \mathbf{x}(0) \in \Re^N$ such that $\mathbf{r}_i\mathbf{x}(k_0)=L_i$ for $i=1,2,\ldots,N-n_d$. As $\sum_{i=1}^{N-n_d} c_{i,n}L_i = L_{N-n_d+n}$ for $n=1,2,\ldots,n_d$, $\sum_{i=1}^{N-n_d} c_{i,n}\mathbf{r}_i\mathbf{x}(k_0)=L_{N-n_d+n}$ for $n=1,2,\ldots,n_d$. Since $\sum_{i=1}^{N-n_d} c_{i,n}\mathbf{r}_i=\mathbf{r}_{N-n_d+n}$ for $n=1,2,\ldots,n_d$, $\mathbf{r}_{N-n_d+n}\mathbf{x}(k_0)=L_{N-n_d+n}$ for $n=1,2,\ldots,n_d$. Hence, $\mathbf{r}_i\mathbf{x}(k_0)=L_i$ for $n=1,2,\ldots,N$. This implies that

$$(\mathbf{I}-\mathbf{A}^P)\mathbf{x}(k_0)=\sum_{j=0}^{P-1}\mathbf{A}^{P-1-j}\mathbf{B}(\mathbf{u}(k_0+j)-\mathbf{s}(k_0+j)). \tag{13}$$

As a result, we have $\mathbf{x}(k_0)=\mathbf{x}(k_0+P)$. Hence, the SDMs exist limit cycle behaviors with period $P$ for $k \ge k_0$. Obviously, $\Psi_P$ is the corresponding nonempty set of initial condition.

When $\Psi_P=\oslash$ or $\exists n \in \{1,2,\ldots,n_d\}$ such that $\sum_{i=1}^{N-n_d} c_{i,n}L_i \neq L_{N-n_d+n}$, then there does not exist $\mathbf{x}(0)$ such that $\mathbf{x}(k_0+P)=\mathbf{x}(k_0)$. Hence, there will not exist any fixed point or periodic state sequence, and this completes the proof ■.

The importance of this Lemma is to characterize the set of initial condition that corresponds to the limit cycle behaviors with period $P$ for $k \ge k_0$. This set of initial condition will be used for the formulation of fuzzy rules shown in Section IV.

This result is a generalization of [2]. In [2], it mainly considers the DC pole cases, that is $k_i=0$ for $i=1,2,\ldots,n_d$. However, we reveal that even though there is not DC pole, but if there exist some poles on the unit circle with their phases are nonzero integer multiple of $(2\pi)/(P)$, then the matrix $\mathbf{Q}$ will also drop rank. Besides, when there are more than one DC poles in the loop filter transfer function, if the degeneracy is equal to the multiplicity of the eigenvalues of matrix $\mathbf{A}$, then the eigen decomposition of matrix $\mathbf{A}$ exists and Lemma 1 is still applied.

As discussed in Sections I and II, stability is an important issue. Hence, the stability analysis is performed before the fuzzy impulsive control strategy is proposed. Although the global stability of the SDMs is usually preferred because global stability implies local stability, sometimes global stability cannot be achieved. Only local stability can be achieved and local stability may be enough for some applications, such as for audio applications [2].

The local stability is discussed as follows. Define the forward and backward dynamics of the system as $\aleph_{\mathrm{f}}:\Re^N \to \Re^N$ and $\aleph_{\mathrm{b}}:\Re^N \to \Re^N$, respectively. That is

$$\begin{aligned}\mathbf{x}(k+1) &\equiv \aleph_{\mathrm{f}}(\mathbf{x}(k)) \text{ in which } \mathbf{x}(k+1)\\ &= \mathbf{Ax}(k)+\mathbf{B}(\bar{\mathbf{u}}-Q(\mathbf{x}(k)))\end{aligned} \tag{14}$$

and

$$\begin{aligned}\mathbf{x}(k-1) &\equiv \aleph_{\mathrm{b}}(\mathbf{x}(k)) \text{ in which } \mathbf{x}(k)\\ &= \mathbf{Ax}(k-1)+\mathbf{B}(\bar{\mathbf{u}}-Q(\mathbf{x}(k-1)))\end{aligned} \tag{15}$$

respectively. Denote

$$x'(k) \equiv b_N\bar{u}+\sum_{i=1}^{N-1}b_{N-i}(\bar{u}-Q(x_i(k)))-\sum_{i=1}^{N}a_{N-i}x_i(k) \tag{16}$$

and

$$\hat{\mathbf{x}}(k) \equiv \left[\frac{x'(k)-Q(x'(k)a_N)b_N}{a_N}, x_1(k),\ldots,x_{N-1}(k)\right]^T. \tag{17}$$

Then

$$\mathbf{A}\hat{\mathbf{x}}(k)+\mathbf{B}(\bar{\mathbf{u}}-Q(\hat{\mathbf{x}}(k)))=\Bigg[x_1(k),\ldots,x_{N-1}(k),x_N(k) + \frac{b_N}{a_0}\left(Q\left(x'(k)a_N\right)-Q\left(\frac{x'(k)-Q\left(x'(k)a_N\right)b_N}{a_N}\right)\right)\Bigg]^T. \tag{18}$$

If $|x'(k)|>|b_N|$, then

$$Q(x'(k)-Q(x'(k)a_N)b_N)=Q(x'(k)). \tag{19}$$

Hence

$$\begin{aligned}Q(x'(k)a_N)-Q\left(\frac{x'(k)-Q(x'(k)a_N)b_N}{a_N}\right)\\ = Q(x'(k)a_N)-Q(x'(k)a_N)=0\end{aligned} \tag{20}$$

and

$$\begin{aligned}\mathbf{A}\hat{\mathbf{x}}(k)+\mathbf{B}(\bar{\mathbf{u}}-Q(\hat{\mathbf{x}}(k)))\\ = [x_1(k),\ldots,x_N(k)]^T=\mathbf{x}(k).\end{aligned} \tag{21}$$

If $|x'(k)|<|b_N|$, then

$$\begin{aligned}Q(x'(k)-Q(x'(k)a_N)b_N)\\ = -Q(x'(k)a_N)Q(b_N)\end{aligned} \tag{22}$$

and

$$Q(x'(k)a_N) - Q\left(\frac{x'(k) - Q(x'(k)a_N)b_N}{a_N}\right) = Q(x'(k)a_N) + Q(x'(k)a_N)Q(a_N b_N). \quad (23)$$

If $Q(a_N b_N) = -1$, then

$$Q(x'(k)a_N) - Q\left(\frac{x'(k) - Q(x'(k)a_N)b_N}{a_N}\right) = 0 \quad (24)$$

and

$$\mathbf{A}\hat{\mathbf{x}}(k) + \mathbf{B}(\bar{\mathbf{u}} - Q(\hat{\mathbf{x}}(k))) = \mathbf{x}(k). \quad (25)$$

Hence, if $|x'(k)| > |b_N|$ or $|x'(k)| < |b_N|$ and $Q(a_N b_N) = -1$, then the backward dynamics of the SDMs can be defined as

$$\aleph_{\mathrm{b}}(\mathbf{x}(k)) = \left[\frac{x'(k) - Q(x'(k)a_N)b_N}{a_N}, x_1(k), \ldots, x_{N-1}(k)\right]^T. \quad (26)$$

Suppose the above conditions are satisfied $\forall k \in Z$. Denote

$$\wp \equiv \{\mathbf{x}(0) : \aleph_{\mathrm{f}}(\mathbf{x}(k)) \in \wp, \text{for } k \geq 0, \text{and } \aleph_{\mathrm{b}}(\mathbf{x}(k)) \in \wp, \text{for } k \leq 0\} \quad (27)$$

and a map $\Im : \wp \to \wp$ such that

$$\Im(\mathbf{x}) \equiv \mathbf{A}\mathbf{x} + \mathbf{B}(\bar{\mathbf{u}} - Q(\mathbf{x})). \quad (28)$$

*Lemma 2:* If $|x'(k)| > |b_N|$ or $|x'(k)| < |b_N|$ and $Q(a_N b_N) = -1$, then $\wp$ is an invariant set under $\Im$. That is, $\Im(\wp) = \wp$. Hence, if $\exists k_0 \in Z$ such that $\mathbf{x}(k_0) \in \wp$, then $\mathbf{x}(k) \in \wp \forall k \in Z$.

*Proof:* The result follows directly from the definition ■.

Although it was reported in [7] that if the invariant set exists and there exists an initial condition in the invariant set, then the local stability is guaranteed. However, the conditions on the existence of the invariant map are not explored and this relationship is explored in Lemma 2.

It is worth noting that if $\exists k_0 \in Z$ such that $\mathbf{x}(k_0) \in \Re^N \backslash \wp$, then $\mathbf{x}(k) \in \Re^N \backslash \wp \forall k \in Z$, and $\mathbf{x}(k)$ may diverge. Hence, it is not sufficient to conclude the global stability of the SDMs.

The importance of Lemma 2 is that it provides information for formulating a fuzzy membership function to achieve local stability.

## IV. Fuzzy Impulsive Control Strategy

### A. Fuzzy Impulsive Control Strategy

Fig. 2 shows the block diagram of how the fuzzy impulsive controller influenced the SDMs. As discussed in Section I, the fuzzy impulsive controller determines the controlled state vectors and reset the state variables of the loop filter to the controlled state variables via a reset circuit. To determine the controlled state vectors, two step procedures are employed. The first step of the procedure is the training phase in which the invariant set and the set of state vectors that exhibits limit cycle behaviors are learnt through training. By generating a set of DC signals inputted to the system with different initial condition, the state vectors are tested if they form an invariant set and exhibit limit cycle behaviors or not. The second step of the procedure is the control phase in which the controlled state vectors are determined and the state variables are reset to the corresponding values. The details are discussed in below.

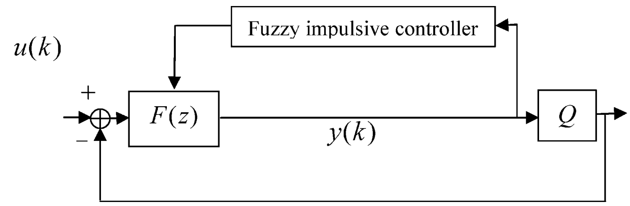


Fig. 2. Block diagram of the interpolative SDM under the fuzzy impulsive control strategy.

As discussed in Section I, we want to minimize the effect of audio clicks. To achieve this goal, we want to minimize the distance between the original state vectors $\mathbf{x}(k_0+1)$ and the controlled state vectors $\mathbf{x}^c(k_0+1)$. However, $\mathbf{x}(k_0+1)$ may be outside the desired bounded region $B_0$, so we define a vector $\mathbf{x}^r \in B_0$ such that $\|\mathbf{x}(k_0+1) - \mathbf{x}^r\|_2$ is minimum and our goal is to minimize the distance between $\mathbf{x}^c(k_0+1)$ and $\mathbf{x}^r$ via a triangular fuzzy membership function as follows:

$$\mu_{\text{continuous}}(\mathbf{x}^c(k_0+1)) \equiv \left(\prod_{i=1}^{N} f_i(\mathbf{x}^c(k_0+1), \mathbf{x}^r)\right)^{\frac{1}{N}} \quad (29)$$

where

$$f_i(\mathbf{x}^c(k_0+1), \mathbf{x}^r) \equiv \begin{cases} \frac{x_i^c(k_0+1) - V_{\mathrm{cc}}}{x_i^r - V_{\mathrm{cc}}}, & x_i^r \leq x_i^c(k_0+1) \leq V_{\mathrm{cc}} \\ \frac{x_i^c(k_0+1) + V_{\mathrm{cc}}}{x_i^r + V_{\mathrm{cc}}}, & -V_{\mathrm{cc}} \leq x_i^c(k_0+1) \leq x_i^r \\ 0, & \text{otherwise.} \end{cases} \quad (30)$$

Since a triangular fuzzy membership function is employed and $\mathbf{x}^r \in B_0, \mu_{\text{continuous}}(\mathbf{x}^c(k_0+1)) = 1$ when $\mathbf{x}^c(k_0+1) = \mathbf{x}^r, \mu_{\text{continuous}}(\mathbf{x}^c(k_0+1)) = 0$ when $\mathbf{x}^c(k_0+1) \in \Re^N \backslash B_0$, and $0 \leq \mu_{\text{continuous}}(\mathbf{x}^c(k_0+1)) \leq 1 \forall \mathbf{x}^c(k_0+1) \in \Re^N$. Hence, $\mu_{\text{continuous}}(\mathbf{x}^c(k_0+1))$ force the new state vectors $\mathbf{x}^c(k_0+1)$ to be within $B_0$. Note that if $\mathbf{x}(k_0+1) \in B_0$, then $\mathbf{x}^r = \mathbf{x}(k_0+1)$ and there will be no audio click effect by setting $\mathbf{x}^c(k_0+1) = \mathbf{x}^r$. Since $\mu_{\text{continuous}}(\mathbf{x}^c(k_0+1))$ captures the knowledge on the closeness between $\mathbf{x}^c(k_0+1)$ and $\mathbf{x}^r$, and the effect of audio clicks is minimized if $\mathbf{x}^c(k_0+1)$ is closed to $\mathbf{x}^r$, this fuzzy membership function can minimize the effect of audio clicks.

As discussed in Sections I and II, the local stability criterion is an important issue. According to Lemma 2, if $|x'(k)| > |b_N|$ or $|x'(k)| < |b_N|$ and $Q(a_N b_N) = -1$, then $\mathbf{x}(k) \in \wp \forall k \in Z$ if $\exists k_0 \in Z$ such that $\mathbf{x}(k_0) \in \wp$. However, the trajectory may not be inside $B_0$ because $\wp$ is usually not equal to $B_0$. In order to guarantee that the trajectory is bounded within $B_0$, we want the controlled state vectors to be inside $\wp \cap B_0$, that is, $\mathbf{x}^c(k_0+1) \in \wp \cap B_0$. Supposing that $\wp \cap B_0 \neq \oslash$. This implies that there exist state vectors that achieve local stability within the set of the desired bounded state variables. Denote $\mathbf{x}^p \in \wp \cap B_0$ such that $\|\mathbf{x}(k_0+1) - \mathbf{x}^p\|_2$ is minimum. If $\wp \cap B_0 \neq \oslash, |x'(k)| > |b_N|$

or $|x'(k)| < |b_N|$ and $Q(a_N b_N) = -1$, then we define the following triangular fuzzy membership function:

$$\mu_{\text{stable}}(\mathbf{x}^c(k_0+1)) \equiv \left(\prod_{i=1}^{N} f_i(\mathbf{x}^c(k_0+1), \mathbf{x}^p)\right)^{\frac{1}{N}}. \quad (31)$$

Since a triangular fuzzy membership function is employed and $\mathbf{x}^p \in B_0, \mu_{\text{stable}}(\mathbf{x}^c(k_0+1)) = 0$ when $\mathbf{x}^c(k_0+1) \in \Re^N \backslash B_0, \mu_{\text{stable}}(\mathbf{x}^c(k_0+1)) = 1$ when $\mathbf{x}^c(k_0+1) = \mathbf{x}^p$ and $0 \leq \mu_{\text{stable}}(\mathbf{x}^c(k_0+1)) \leq 1 \forall \mathbf{x}^c(k_0+1) \in \Re^N$. Hence, $\mu_{\text{stable}}(\mathbf{x}^c(k_0+1))$ force the new state vectors $\mathbf{x}^c(k_0+1)$ to be within $B_0$. If $\mathbf{x}(k_0+1) \in \wp \cap B_0$, then $\mathbf{x}^p = \mathbf{x}(k_0+1)$. By setting $\mathbf{x}^c(k_0+1) = \mathbf{x}^p$, the local stability criterion is satisfied. Since $\mu_{\text{stable}}(\mathbf{x}^c(k_0+1))$ captures the knowledge on the closeness between $\mathbf{x}^c(k_0+1)$ and $\mathbf{x}^p$, which also reflects the closeness between $\mathbf{x}^c(k_0+1)$ and the set of state vectors that achieved local stability within the desired bounded region, this fuzzy membership function can capture the local stability criterion into the system.

However, if $\wp \cap B_0 = \oslash$, then $\mathbf{x}^p$ does not exist. Or if $\exists k' \in Z$ such that $|x'(k)| < |b_N|$ and $Q(a_N b_N) = 1$, then the local stability criterion is not guaranteed. In this case, the SDM may suffer from an instability problem. In order to avoid this case to be happened, if $\wp \cap B_0 = \oslash$, or if $\exists k' \in Z$ such that $|x'(k)| < |b_N|$ and $Q(a_N b_N) = 1$, then we define

$$\mu_{\text{stable}}(\mathbf{x}^c(k_0+1)) \equiv \begin{cases} \delta_{\text{stable}}, & \mathbf{x}^c(k_0+1) \in B_0 \\ 0, & \mathbf{x}^c(k_0+1) \in \Re^N \backslash B_0 \end{cases} \quad (32)$$

where $1 \geq \delta_{\text{stable}} > 0$ and $\delta_{\text{stable}}$ is very closed to zero. The reasons why small value of $\delta_{\text{stable}}$ can avoid the instability problem are discussed in Section IV.B. Since the fuzzy membership value of the state vectors outside $B_0$ is exactly equal to zero, this fuzzy membership function will force the new state vectors $\mathbf{x}^c(k_0+1)$ to be within $B_0$.

As discussing in Section I, the occurrence of limit cycle behaviors should be avoided. Since $\bigcup_{\forall P>0} \Psi_P$ is the set of state vectors that exhibiting limit cycle behavior, we do not want to move the new state vectors $\mathbf{x}^c(k_0+1)$ into $\bigcup_{\forall P>0} \Psi_P$. Moreover, we do not want to move $\mathbf{x}^c(k_0+1)$ into $\bigcup_{\forall k \leq k_0} \{\mathbf{x}(k)\}$ too. This is because after a certain number of iterations, the state vectors may go to the same points in the state space and cause limit cycle behaviors to occur. Define

$$\text{PER}(k_0) \equiv \left(\bigcup_{\forall P>0} \Psi_P\right) \cup \left(\bigcup_{\forall k \leq k_0} \{\mathbf{x}(k)\}\right). \quad (33)$$

If $\text{PER}(k_0) \cap B_0 = B_0$, then all the state vectors in $B_0$ may result limit cycle behaviors and this situation should be avoided. On the other hand, if $\text{PER}(k_0) \cap B_0 = \oslash$, then we cannot find a state vector $\mathbf{x}^q \in B_0 \cap \text{PER}(k_0)$ such that $\|\mathbf{x}(k_0+1) - \mathbf{x}^q\|_2$ is minimum. Hence, if $\text{PER}(k_0) \cap B_0 = B_0$ or $\text{PER}(k_0) \cap B_0 = \oslash$, we define the fuzzy membership function as

$$\mu_{\text{aperiodic}}(\mathbf{x}^c(k_0+1)) \equiv \begin{cases} \delta_{\text{aperiodic}}, & \mathbf{x}^c(k_0+1) \in B_0 \\ 0, & \mathbf{x}^c(k_0+1) \in \Re^N \backslash B_0 \end{cases} \quad (34)$$

where $1 \geq \delta_{\text{aperiodic}} > 0$ and $\delta_{\text{aperiodic}}$ is also very closed to zero. Similarly, the reason why small value of $\delta_{\text{aperiodic}}$ can avoid the occurrence of limit cycle behaviors is discussed in Section IV.B. Otherwise, we define the fuzzy membership function as shown in (35) at the bottom of the page. Since $f_i$ is a triangular fuzzy membership function and $\mathbf{x}^q \in B_0, \mu_{\text{aperiodic}}(\mathbf{x}^c(k_0+1)) = 0$ when $\mathbf{x}(k_0+1) \in B_0 \cap \text{PER}(k_0)$ because $\mathbf{x}^q = \mathbf{x}(k_0+1)$ when $\mathbf{x}(k_0+1) \in B_0 \cap \text{PER}(k_0), \mu_{\text{aperiodic}}(\mathbf{x}^c(k_0+1)) = 0$ when $\mathbf{x}^c(k_0+1) \in \Re^N \backslash B_0$ and $0 \leq \mu_{\text{aperiodic}}(\mathbf{x}^c(k_0+1)) \leq 1 \forall \mathbf{x}^c(k_0+1) \in \Re^N$. Hence, $\mu_{\text{aperiodic}}(\mathbf{x}^c(k_0+1))$ force the new state vectors $\mathbf{x}^c(k_0+1)$ to be within $B_0$. Since $\mu_{\text{aperiodic}}(\mathbf{x}^c(k_0+1))$ captures the knowledge on the separation between $\mathbf{x}^c(k_0+1)$ and $B_0 \cap \text{PER}(k_0)$, which also reflects the separation between $\mathbf{x}^c(k_0+1)$ and the set of state vectors within the desired bounded region that exhibits limit cycle behaviors, $\mu_{\text{aperiodic}}(\mathbf{x}^c(k_0+1))$ can be used to avoid the occurrence of limit cycle behaviors.

Once the fuzzy membership functions are defined, we can define the fuzzy impulsive control law as follows.

If $\mathbf{Ax}(k_0) + \mathbf{B}(\bar{\mathbf{u}} - Q(\mathbf{x}(k_0))) \in \Re^N \backslash B_0$, then the fuzzy impulsive controller will reset the state variables of the loop filter to $\mathbf{x}^c(k_0+1)$ where $\mathbf{x}^c(k_0+1)$ is the state vector such that the following function is maximized:

$$\mu_{\mathbf{x}^c(k_0+1)}(\mathbf{x}^c(k_0+1)) \equiv \max_{\mathbf{x}^c(k_0+1) \in \Re^N} (\mu_{\text{stable}}(\mathbf{x}^c(k_0+1)) \mu_{\text{aperiodic}} (\mathbf{x}^c(k_0+1)) \mu_{\text{continuous}}(\mathbf{x}^c(k_0+1)))^{1/3}. \quad (36)$$

Otherwise, no control force is applied to the SDMs.

*Lemma 3:* $\forall \bar{u} \in \Re$, $\forall \mathbf{x}(0) \in \Re^N$, $\forall a_i \in \Re$ for $i = 0, 1, \ldots, N$ and $\forall b_j \in \Re$ for $j = 1, \ldots, N, \mathbf{x}^c(k) \in B_0$ for $k > 0$.

*Proof:* It can be seen that $\forall \bar{u} \in \Re, \forall \mathbf{x}(0) \in \Re^N, \forall a_i \in \Re$ for $i = 0, 1, \ldots, N, \forall b_j \in \Re$ for $j = 1, \ldots, N, \forall k_0 \geq 0$ and $\forall \mathbf{x}^c(k_0+1) \in B_0, \mu_{\text{continuous}}(\mathbf{x}^c(k_0+1)) > 0$ and $\mu_{\text{stable}}(\mathbf{x}^c(k_0+1)) > 0$. If $\text{PER}(k_0) \cap B_0 = B_0$ or $\text{PER}(k_0) \cap B_0 = \oslash$, then $\mu_{\text{aperiodic}}(\mathbf{x}^c(k_0+1)) > 0$. Although $\mu_{\text{aperiodic}}(\mathbf{x}^c(k_0+1)) = 0$ if $\text{PER}(k_0) \cap B_0 \neq B_0, \text{PER}(k_0) \cap B_0 \neq \oslash$, and $\mathbf{x}(k_0+1) \in B_0 \cap \text{PER}(k_0)$, since $\text{PER}(k_0) \cap B_0 \neq B_0, \exists \mathbf{x}^c(k_0+1) \in B_0 \backslash \text{PER}(k_0)$

$$\mu_{\text{aperiodic}}(\mathbf{x}^c(k_0+1)) \equiv \begin{cases} 1 - \left(\prod_{i=1}^{N} f_i\left(\mathbf{x}^c(k_0+1), \mathbf{x}^q\right)\right)^{\frac{1}{N}}, & \mathbf{x}^c(k_0+1) \in B_0 \\ 0, & \mathbf{x}^c(k_0+1) \in \Re^N \backslash B_0 \end{cases} \quad (35)$$

such that $\mu_{\mathrm{aperiodic}}(\mathbf{x}^c(k_0+1)) > 0$. Hence, $\exists \mathbf{x}^c(k_0+1) \in B_0 \backslash \mathrm{PER}(k_0)$ such that $\mu_{\mathbf{x}^c(k_0+1)}(\mathbf{x}^c(k_0+1)) > 0$. As a result, if $\mathbf{A}\mathbf{x}(k_0) + \mathbf{B}(\bar{\mathbf{u}} - Q(\mathbf{x}(k_0))) \in \Re^N \backslash B_0$, then the fuzzy impulsive controller will reset the state vector of the loop filter to $\mathbf{x}^c(k_0+1)$ where $\mathbf{x}^c(k_0+1) \in B_0 \backslash \mathrm{PER}(k_0)$. If $\mathbf{A}\mathbf{x}(k_0) + \mathbf{B}(\bar{\mathbf{u}} - Q(\mathbf{x}(k_0))) \in B_0$, since no control force is applied to the SDM, $\mathbf{x}^c(k_0+1) = \mathbf{x}(k_0+1) \in B_0$. Hence, $\mathbf{x}^c(k) \in B_0$ for $k > k_0$. Thus, $\forall k_0 \geq 0, \mathbf{x}^c(k) \in B_0$ for $k > 0$. And this completes the proof ■.

It is worth noting that different values of $\bar{u}, \mathbf{x}(0), a_i$ for $i = 0, 1, \ldots, N$ and $b_j$ for $j = 1, \ldots, N$, will affect the existence of $\wp$ and $\bigcup_{\forall P>0} \Psi_P$. However, Lemma 3 is still applied even though $\wp = \oslash$ or $\wp = B_0$, and $\bigcup_{\forall P>0} \Psi_P = \oslash$ or $\bigcup_{\forall P>0} \Psi_P = B_0$. Hence, Lemma 3 guarantees that the controlled trajectory is bounded within $B_0$ no matter what the input step size, the initial condition and the filter parameters are. This is very important because we do not want the trajectory of the SDM to be unbounded if the input step size is increased, or the initial condition or the loop filter of the SDMs are changed. Another advantage of this fuzzy impulsive control strategy is that we can alter the maximum bound of the state variables easily by setting the value of $V_{\mathrm{cc}}$ appropriately, which is independent of the input step size, the initial condition and the filter parameters.

*Lemma 4:* $\forall \bar{u} \in \Re$, $\forall \mathbf{x}(0) \in \Re^N$, $\forall a_i \in \Re$ for $i = 0, 1, \ldots, N$ and $\forall b_j \in \Re$ for $j = 1, \ldots, N, \|\mathbf{x}^c(k+1) - \mathbf{x}^r\|_2 \leq 2V_{\mathrm{cc}}\sqrt{N}$ for $k > 0$.

*Proof:* Since $\forall \bar{u} \in \Re, \forall \mathbf{x}(0) \in \Re^N, \forall a_i \in \Re$ for $i = 0, 1, \ldots, N$ and $\forall b_j \in \Re$ for $j = 1, \ldots, N, \mathbf{x}^c(k) \in B_0$ for $k > 0$, the result follows directly ■.

The importance of this Lemma is that it guarantees the norm of the difference between $\mathbf{x}^r$ and $\mathbf{x}^c(k+1)$ being bounded by $2V_{\mathrm{cc}}\sqrt{N}$, no matter what the input step size, the initial condition and the filter parameters are. As discussed in above, we do not want the norm of the difference between $\mathbf{x}^r$ and $\mathbf{x}^c(k+1)$ to be too large because the effect of audio clicks may be too large for these situations.

*Lemma 5:* If $\exists k_0 \in Z$ such that $\mathrm{PER}(k) \cap B_0 \neq B_0$ for $k \geq k_0$, and $\mathbf{A}\mathbf{x}(k_0) + \mathbf{B}(\bar{\mathbf{u}} - Q(\mathbf{x}(k_0))) \in \Re^N \backslash B_0$, then $\nexists M > 0$ such that $\mathbf{x}^c(k) = \mathbf{x}^c(k+M)$ for $k > k_0$.

*Proof:* The proof follows directly from Lemma 3 ■.

The importance of this Lemma is that it states the condition that limit cycle behaviors do not occur when the fuzzy impulsive control strategy is applied at once. We will show the contrast in Section V that the clipping control strategy usually results in the limit cycle behaviors, while our approach will minimize the occurrence of limit cycle behaviors.

### *B. Parameters in the Fuzzy Impulsive Controller*

There are only three parameters in the fuzzy impulsive control strategy. They are $V_{\mathrm{cc}}, \delta_{\mathrm{aperiodic}}$, and $\delta_{\mathrm{stable}}$. $V_{\mathrm{cc}}$ is the maximum allowable bound on each state variable and this value is determined based on the real situations, such as the hardware constraints and the safety specifications, etc. For example, if the hardware operates normally in a safe condition only when the state variables are bounded by 20 V, then $V_{\mathrm{cc}}$ may be set accordingly. For the parameters $\delta_{\mathrm{aperiodic}}$ and $\delta_{\mathrm{stable}}$, the fuzzy impulsive controller works properly $\forall \delta_{\mathrm{aperiodic}} \in (0, 1]$ and $\forall \delta_{\mathrm{stable}} \in (0, 1]$. However, since $\delta_{\mathrm{aperiodic}}$ represents the fuzzy membership value of how to avoid the occurrence of limit cycle at $\mathbf{x}^c(k_0+1)$ when $\mathrm{PER}(k_0) \cap B_0 = B_0$ or $\mathrm{PER}(k_0) \cap B_0 = \oslash$, and all the state vectors in $B_0$ may cause the trajectory to exhibit limit cycle behaviors if $\mathrm{PER}(k_0) \cap B_0 = B_0$, we suggest the SDM control designers to set this value as a small number closed to zero, such as $10^{-3}$. For $\delta_{\mathrm{stable}}$, since it represents the fuzzy membership value of the local stability of the SDM at $\mathbf{x}^c(k_0+1)$ when $\wp \cap B_0 = \oslash$, or if $\exists k' \in Z$ such that $|x'(k)| < |b_N|$ and $Q(a_N b_N) = 1$, and in this case, the SDM may exhibit divergent behavior if the fuzzy impulsive control strategy is not applied, we recommend the SDM control designers to set this value as a small number closed to zero too, for example, $10^{-3}$.

### *C. Complexity Issue*

Although more fuzzy rules and sophisticated fuzzy engine will improve the performance of the SDMs, this will increase the complexity of the system and may cause real time processing problems, particular for audio applications [2]. The Nyquist rate for audio signal is 44.1 kHz [2], since the input signals are typically oversampled at 64 or 128 [2], the number of samples inputted to the SDM per second is 2.8224 M or 5.6448 M. Because several megasamples are needed to process per second, only three basic fuzzy rules are captured and only a simple fuzzy engine is used to reduce the complexity for processing. According to the simulation results shown in Section V, these three basic rules and a simple fuzzy engine is enough for achieving the objectives.

### *D. Implementation of the Fuzzy Impulsive Controller*

As discussed in the above, the fuzzy impulsive controller resets the state variables of the loop filter to the controlled state variables of $\mathbf{x}^c(k_0+1)$ if $\mathbf{A}\mathbf{x}(k_0) + \mathbf{B}(\bar{\mathbf{u}} - Q(\mathbf{x}(k_0))) \in \Re^N \backslash B_0$, and $\mathbf{x}^c(k_0+1)$ is calculated based on (36). Numerical solvers, such as MATLAB or MATCAD, can be employed for solving (36). To reset the state variables of the loop filter, many existing reset circuits can be employed [8].

## V. Simulation Results

To illustrate our results, a fifth-order SDM with the loop filter transfer function is illustrated in (37) at the bottom of the page. This fifth-order SDM is commonly employed in the industry [2].

$$\frac{20z^{-1} - 74z^{-2} + 103.0497z^{-3} - 64.0015z^{-4} + 14.9584z^{-5}}{1 - 5z^{-1} + 10.0025z^{-2} - 10.0075z^{-3} + 5.0075z^{-4} - 1.0025z^{-5}} \tag{37}$$

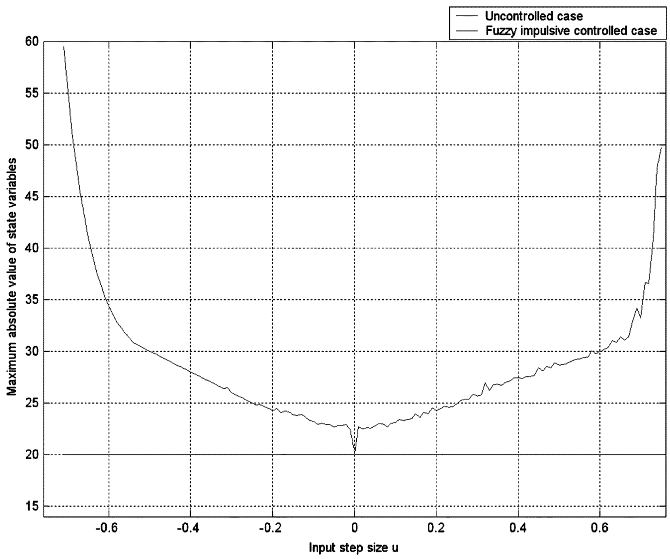


Fig. 3. Plot of the maximum absolute value of the state variables (realized in direct form) against the input step size when $\tilde{\mathbf{x}}(0) = [0,\ 0,\ 0,\ 0,\ 0]^T$.

The SDM can be implemented via the Jordan form [2] and can be realized as the following state space equation:

$$\tilde{\mathbf{x}}(k+1) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{B}}(u(k) - y(k)) \tag{38}$$

for $k \geq 0$, where

$$y(k) = Q(\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k)) \tag{39}$$

$$\tilde{\mathbf{A}} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -0.0018 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -0.000685 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\tilde{\mathbf{B}} \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{\mathbf{C}} \equiv \begin{bmatrix} 20 \\ 6 \\ 1 \\ 0.09375 \\ 0.00589 \end{bmatrix}^T. \tag{40}$$

Assume that the initial condition of this SDM is zero, that is, $\tilde{\mathbf{x}}(0) = [0,\ 0,\ 0,\ 0,\ 0]^T$. By using a simple transformation, this SDM can be realized by the direct form and the corresponding initial condition is $\mathbf{x}(0) = [0,\ -5,\ 28.5,\ 32.25,\ 35.9793]^T$ when $u = 0.75$. We can check that the trajectory of this SDM is bounded for this initial condition ($\tilde{\mathbf{x}}(0) = [0,\ 0,\ 0,\ 0,\ 0]^T$) if the input step size is approximately between $-0.71$ and $0.75$, and may diverge if the input step size is outside this range. The relationship between the maximum absolute value of the state variables (realized in the direct form) and the input step size is plotted in Fig. 3. From the simulation result, we can see that even though the trajectory is bounded for this range of input step size, the maximum absolute value of the state variables is between 20.0523 and 59.4633, which may be too large for some practical applications [2]. Fig. 3 also shows the plot of the maximum absolute value of the state variables (also realized in the direct form) for $k > 0$ versus the input step size when the fuzzy impulsive control strategy is applied at $V_{\mathrm{cc}} = 20$. According to Lemma 3, the maximum absolute value of the state variables of the controlled SDM is bounded by $V_{\mathrm{cc}}$ for $k > 0$ and $\forall \bar{u} \in \Re$, even though $|\bar{u}| \geq V_{\mathrm{cc}}$. Hence, we can guarantee that the state variables are bounded by 20.

This SDM is not globally stable. That means, $\exists \tilde{\mathbf{x}}(0) \in \Re^N$ such that the trajectory is unbounded. For example, when $\bar{u} = 0.75$, Fig. 4(a) and (b) shows the responses of $x_1(k)$ with $\tilde{\mathbf{x}}(0) = [0,\ 0,\ 0,\ 0,\ 0]^T$ and $\tilde{\mathbf{x}}(0) = [0.001,\ 0,\ 0,\ 0,\ 0]^T$, respectively. It can be seen from Fig. 4(a) and (b) that even though the SDM exhibits acceptable behavior when $\tilde{\mathbf{x}}(0) = [0,\ 0,\ 0,\ 0,\ 0]^T$ and the difference between these two initial conditions is very small, the SDM exhibits divergent behavior when $\tilde{\mathbf{x}}(0) = [0.001,\ 0,\ 0,\ 0,\ 0]^T$ and the behaviors of the SDM for these two different initial conditions are very different. On the other hand, according to Lemma 3, the maximum absolute value of the state variables is always bounded by $V_{\mathrm{cc}}$ for $k > 0$ and $\forall \mathbf{x}(0) \in \Re^N$ if the fuzzy impulsive control strategy is applied. Fig. 4(c) and (d) show the corresponding state responses when the fuzzy impulsive control strategy is applied at $V_{\mathrm{cc}} = 40$. From the simulation result, we see that the SDM exhibits acceptable behavior with the state variables bounded by $V_{\mathrm{cc}}$ for both of these two initial conditions.

For comparison with other control strategies, consider the time delay feedback control strategy proposed in [6], in which the controller is in the form of $-K_{\mathrm{c}}(1 - z^{-1})$. Denote $\lambda_i$ for $i = 1, 2, \ldots, 6$ be the poles of the effective loop filter. Since $\lambda_i$ for $i = 1, 2, \ldots, 6$ depends on the value of $K_{\mathrm{c}}$, it can be shown that $\max_{i=1,2,\ldots,6} |\lambda_i| > 1\, \forall K_{\mathrm{c}} \in \Re$ and the minimum value of $\max_{i=1,2,\ldots,6} |\lambda_i|$ occurs at $K_{\mathrm{c}} = 0$. When $K_{\mathrm{c}} = 0$, it reduces to the uncontrolled case. By selecting a value of $K_{\mathrm{c}}$ which is very closed to zero, for example $K_{\mathrm{c}} = 2 \times 10^{-5}$, and setting the initial condition and the input step size as the previous values, that is, $\mathbf{x}(0) = [0,\ -5,\ 28.5,\ 32.25,\ 35.9793,\ 39.5612]^T$ and $\bar{u} = 0.75$ (the initial condition is determined based on zero initial condition of the Jordan form), it is found that the trajectory diverges as shown in Fig. 5. Hence, the time delay feedback control strategy fails to stabilize this SDM.

To compare the fuzzy impulsive control strategy to the clipping control strategy, that is, set $x_i(k) = V_{\mathrm{cc}}Q(x_i(k))$ if $|x_i(k)| \geq V_{\mathrm{cc}}$ for $i = 1, 2, \ldots, N$, it is found that limit cycle behaviors may occur if the clipping control strategy is applied. Fig. 6 shows the magnitude response of $s(k)$ when $\bar{u} = 0.75, \tilde{\mathbf{x}}(0) = [0,\ 0,\ 0,\ 0,\ 0]^T$ and the clipped level is set at 40. It can be seen from Fig. 6 that there is an impulse located at $(\pi/2)$ if the clipping control strategy is applied, which demonstrates that the SDM exhibits a limit cycle with period 2. On the other hand, the spectrum is quite flat for the SDM when the fuzzy impulsive control strategy is applied with $V_{\mathrm{cc}} = 40$, which demonstrates that the SDM exhibits acceptable behavior and the limit cycle behavior is avoided.

Fig. 7 shows the SNR of SDMs under the clipping control strategy with the clipped level set at 28. SNR is calculated

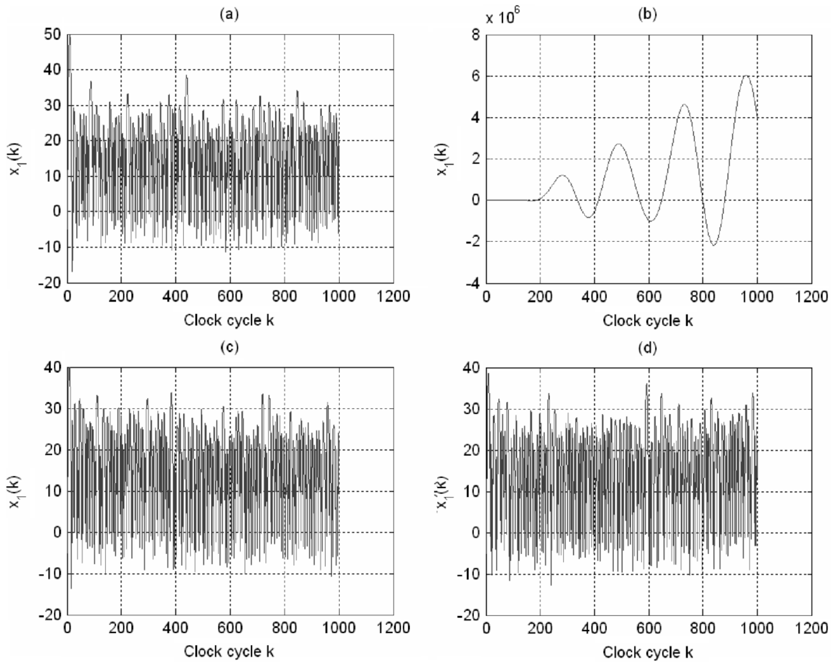


Fig. 4. Response of $x_1(k)$ when $\bar{u} = 0.75$ and (a) initial condition $\tilde{\mathbf{x}}(0) = [0,\ 0,\ 0,\ 0,\ 0]^T$ when no control strategy is applied, (b) initial condition $\tilde{\mathbf{x}}(0) = [0.001,\ 0,\ 0,\ 0,\ 0]^T$ when no control strategy is applied, (c) initial condition $\tilde{\mathbf{x}}(0) = [0,\ 0,\ 0,\ 0,\ 0]^T$ when the fuzzy impulsive control strategy with $V_{\rm cc} = 40$ is applied, and (d) initial condition $\tilde{\mathbf{x}}(0) = [0.001,\ 0,\ 0,\ 0,\ 0]^T$ when the fuzzy impulsive control strategy with $V_{\rm cc} = 40$ is applied.

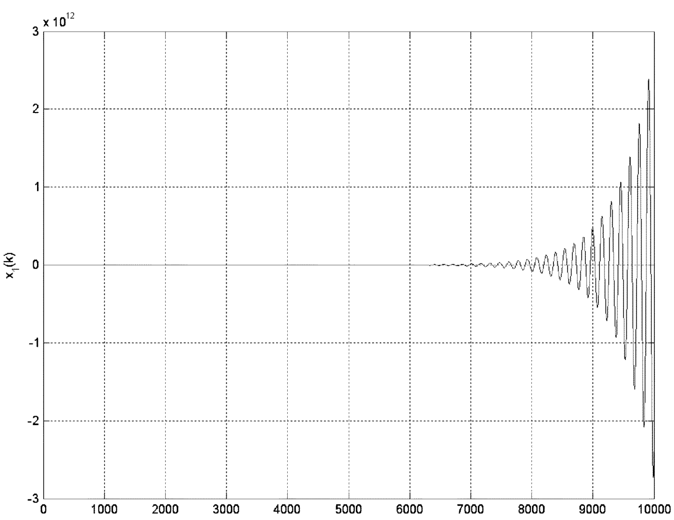


Fig. 5. Response of $x_1(k)$ with input step size $\bar{u} = 0.75$ and initial condition $\mathbf{x}(0) = [0,\ -5,\ 28.5,\ 32.25,\ 35.9793,\ 39.5612]^T$ when the time delay feedback control strategy with $K_{\rm c} = 2 \times 10^{-5}$ is applied.

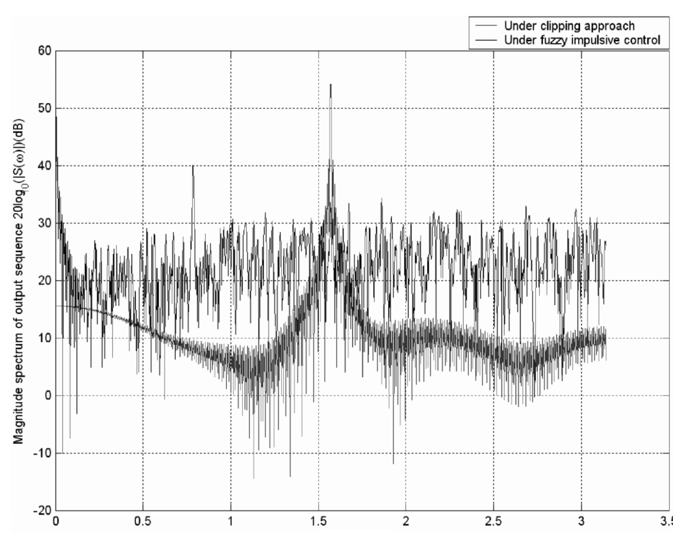


Fig. 6. Magnitude response of the output sequence when $\bar{u} = 0.75$ and initial condition $\tilde{\mathbf{x}}(0) = [0,\ 0,\ 0,\ 0,\ 0]^T$ for both the clipping and fuzzy impulsive control strategies are applied with the state variables bounded by 40.

using [9], where the frequency of the input sinusoidal signals is $(2/3)$ of the passband bandwidth. The oversampling ratio is 64, and initial conditions are given by $\tilde{\mathbf{x}}(0) = [0,\ 0,\ 0,\ 0,\ 0]^T$. It can be seen from Fig. 7 that the SNR of both the clipping and fuzzy impulsive control strategies with the state variables bounded by 28 are the same when the input magnitude is less than 0.52. This is because both the maximum absolute value of the state variables (realized in the direct form) do not exceed 28 in this input magnitude range. However, if the input magnitude exceeds this range, the SNR corresponding to the clipping control strategy may drop to less than 1.2562 dB because of the occurrence of limit cycle behaviors. On the other hand, the SDM performs normally under the fuzzy impulsive control strategy. Hence, the SNR of the SDM under the fuzzy impulsive control strategy has an average of 41.8281 dB improvement compared to the clipping control strategy for outside this input magnitude range.

It can be seen from Fig. 8 that the probability of the control force to be applied by the fuzzy impulsive control strategy is 0.0135 for the input magnitude range greater than or equal to

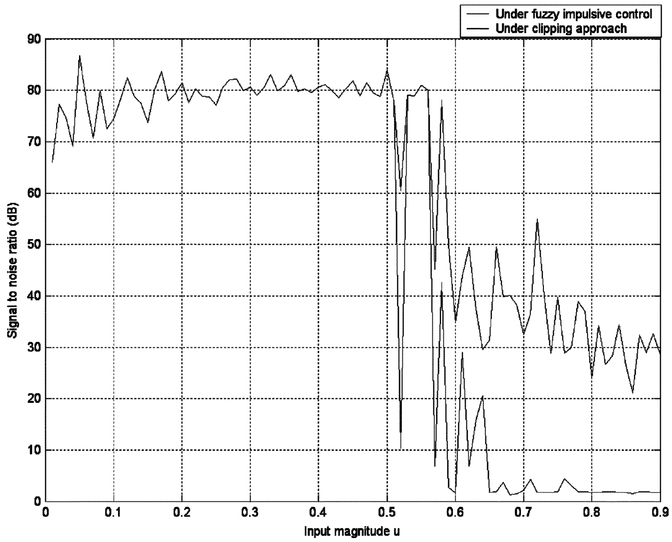


Fig. 7. SNR of SDMs when input sinusoidal frequency is $(2/3)$ of the passband bandwidth, initial condition $\tilde{\mathbf{x}}(0) = [0,\ 0,\ 0,\ 0,\ 0]^T$ and the state variables are bounded by 28.

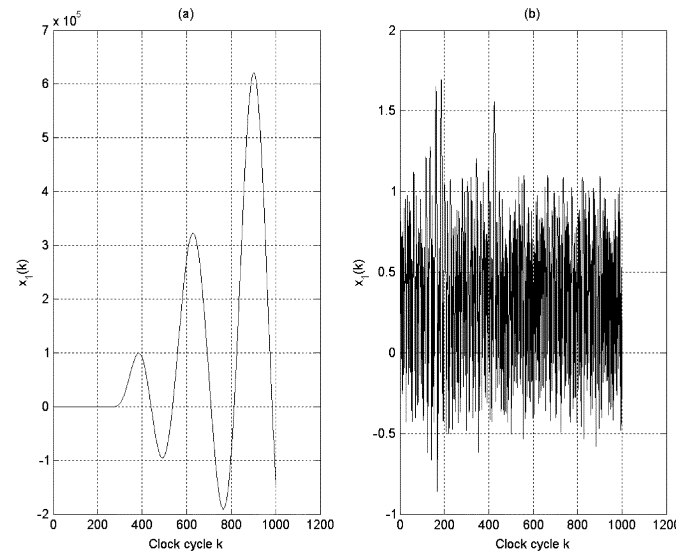


Fig. 9. Response of $x_1(k)$ with initial condition $\tilde{\mathbf{x}}(0) = [0,\ 0,\ 0,\ 0,\ 0]^T$ and input step size $\bar{u} = 0.59$ (a) when no control strategy is applied. (b) when the fuzzy impulsive control strategy with $V_{\rm cc} = 2$ is applied.

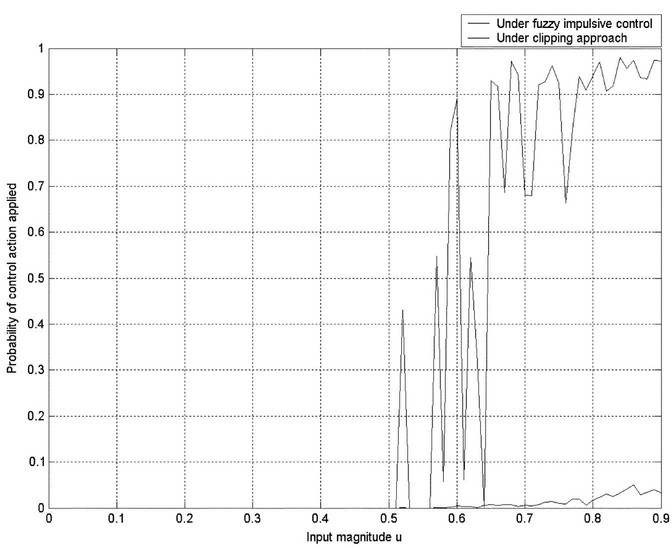


Fig. 8. Probability of control force applied to the SDM when the input sinusoidal frequency is $(2/3)$ of the passband bandwidth, initial condition $\tilde{\mathbf{x}}(0) = [0,\ 0,\ 0,\ 0,\ 0]^T$ and the state variables are bounded by 28.

0.52, as opposed to a probability of 0.6926 for the clipping control strategy. Hence, the number of reset action on the state variables of the loop filter is much reduced when applying fuzzy impulsive control strategy. This is because the fuzzy impulsive control strategy tends to reset the state vectors inside the invariant set if it exists and the state vectors will tend to stay inside the invariant set without applying control force again soon afterwards. This demonstrates that the fuzzy impulsive control strategy is more efficient than the clipping control strategy.

To verify the independence of the filter parameters on the fuzzy impulsive control strategy, consider another fifth-order SDM with the transfer function [2] shown in (41) at the bottom of the page. $u(k)$ This SDM is also widely used in the industry [2]. The trajectory of this SDM with $\bar{u} = 0.59$ and $\tilde{\mathbf{x}}(0) = [0,\ 0,\ 0,\ 0,\ 0]^T$ is shown in Fig. 9(a), and it can be seen from Fig. 9(a) that the trajectory diverges. On the other hand, when the fuzzy impulsive control strategy is applied with $V_{\rm cc} = 2$, according to Lemma 3, the maximum absolute value of the state variables (realized in the direct form) is always bounded by $V_{\rm cc}$ for $k > 0, \forall a_i \in \Re$ for $i = 0, 1, \ldots, N$ and $\forall b_j \in \Re$ for $j = 1, \ldots, N$, as shown in Fig. 9(b).

## VI. Conclusion

In this paper, we have proposed the fuzzy impulsive control strategy for the stabilization of high-order interpolative SDMs in which the occurrence of limit cycle behaviors and the effect of audio clicks are minimized. Since the effective poles of the loop filter are not affected by the control strategy, the SNR performance of the SDMs is maintained or improved after control. Moreover, the controlled trajectory is guaranteed to be bounded no matter what the input step size, the initial condition and the filter parameters are. Comparisons between the fuzzy impulsive control strategy and some existing control strategies show that the fuzzy impulsive control strategy is much effective in terms of producing much higher SNR and efficient in terms of requiring less control force applied to the system.

$$\frac{0.7919z^{-1} - 2.8630z^{-2} + 3.9094z^{-3} - 2.3873z^{-4} + 0.5498z^{-5}}{1 - 5z^{-1} + 10.0023z^{-2} - 10.0069z^{-3} + 5.0069z^{-4} - 1.0023z^{-5}} \tag{41}$$

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