Description of Limit Cycles in Sigma–Delta Modulators

Derk Reefman, Josh Reiss, Erwin Janssen, and Mark Sandler

***Abstract*—A mathematical framework, based on state-space modeling, for the description of limit cycles (LCs) of 1-bit sigma–delta modulators (SDMs) is presented. It is proved that periodicity in bit output pattern of the SDM implies a periodic orbit in state-space variables. While the state-space description is generally applicable for periodic inputs, the focus is on dc inputs, since this represents the most relevant practical condition. An outcome of the analysis is that, in general, for an Nth-order SDM, at least $N-1$ initial conditions need to be fixed in order to have LC behavior. Expressions for the minimum disturbance of the input or initial conditions that is needed to break up a LC are also presented. Special focus is given to the case where the disturbance takes the form of "dithering the quantizer", and it is shown that this form of dither is a suboptimal approach to remove LCs. The stability of LCs is determined, and it is demonstrated that a resonator section, as often employed to increase the dynamic range of an SDM, has an adverse effect on LC behavior in that it stabilizes LCs. Furthermore, the experimental observation that high order SDMs are less susceptible to LCs is underpinned. Finally, some examples are provided which illustrate the theoretical results.**

***Index Terms*—Analog–digital (A/D) conversion, digital–analog (D/A) conversion, limit cycles (LCs), nonlinear circuits, nonlinear systems, sigma–delta modulation.**

I. Introduction

SIGMA–DELTA modulation is an increasingly popular technique for coding data streams [1]. The technique has provided powerful means for converting analog to digital signals and *vice versa* with low circuit complexity and large robustness against circuit imperfections. As a result of this, 1-bit sigma–delta based analog-to-digital (A/D) and digital-to-analog (D/A) converters are widely used in audio applications, such as cellular phone technology and high-end stereo systems. In particular, it has seen a further boost in interest due to the introduction of super audio compact disk (SA-CD). SA-CD is based on a digital format coined DSD, which is a 1-bit coded representation of the audio stream with a sample rate of 2.8 MHz. In the SA-CD standard, no reference is made to the technique which is employed to create such a 1-bit stream, yet sigma–delta modulation is clearly one of the techniques that is capable of creating a high-quality 1-bit stream. Sigma–delta modulation, originally conceived by de Jager [2], is a well-established technique. However, theoretical understanding of the concept is very limited [1]. The most important progress in the description of SDMs is reported in the work of Risbo [3] and Hein and Zakhor [4], while a very useful linearization technique is described in [5] and further elaborated on by Magrath [6]. Yet in all these developments, there is no unified description of SDMs. Instead, several models are provided, each of which describes some aspects of an SDM to a certain accuracy. Though it is not the intention of this paper to give a comprehensive theory of sigma–delta modulation in general, it does give an exact mathematical framework. Within this framework, some aspects of sigma–delta modulation can be quantitatively understood. In this paper, the focus is on the characterization of limit cycles (LCs). The analysis will be restricted to the class of "feedforward SDMs" (also called "interpolative SDMs") [1], which represents a class of often used SDM topologies. The analysis can be easily adapted to deal with other topologies as well [7].

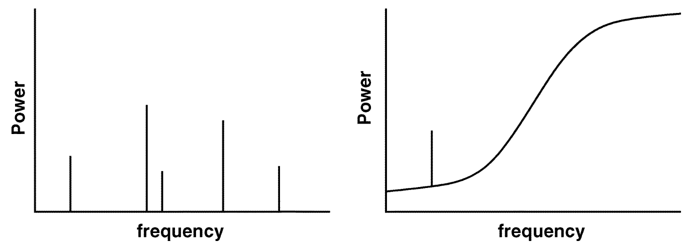

Fig. 1. Illustration of the definitions used in the paper to distinguish a LC (left) from an idle tone (right). An LC consists of a finite number of discrete peaks in the frequency spectrum; an idle tone is a peak in the frequency spectrum, but superposed on a noise background.

We use the definition of an LC which is customary in the world of sigma–delta design engineers. An LC is a sequence of P output bits, which repeats itself indefinitely. As such, we want to contrast this definition with that of an "idle tone," which is a discrete peak in the frequency spectrum of the output of an SDM, but superposed on a background of noise (see Fig. 1). Hence, in that case, there is no unique series of P bits which repeats itself.

Fundamental work on LCs in SDMs has usually been constrained to low order SDMs [8]–[10], and hence is of little practical value to engineers who use high order noise shaping techniques. The basis for the approach used in this paper has been provided in previous work, most notably in [4] and [11]. An approach which bears some resemblance to the one presented in the current paper is discussed in [12]. In [12], however, the relative frequency content of possible LCs in some specific SDMs is discussed. We derive results for a general SDM and focus on the character of the LCs, and their stability in particular. An important assumption in this earlier work is that a periodic bit output pattern implies a

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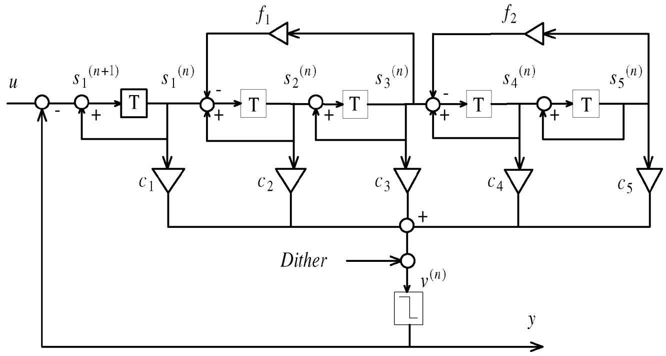

Fig. 2. States in fifth-order SDM.

periodic orbit in state-space variables. This assumption is not obvious, but turns out to be true as will be proven in this paper. Based on a state-space description, we will present an exact description of LCs in SDMs. While drawing on some known results from linear algebra, some remarkable results for LCs in SDMs are obtained, such as the persistance of LCs while dithering the SDM. Although in its pure definition, a LC is a periodic pattern of infinite duration, in practical situations finite duration periodic sequences can be equally annoying. The finite duration patterns touch upon the important subject of stability of a LC. How much time it takes until a small perturbation moves a LC out of its periodic pattern.

The paper is organized as follows. In Section II, the mathematical framework, based on a state-space description of the SDM, is presented. All the following chapters are based on this formulation. In Section III, this formulation is applied to practical SDM designs. Some basic quantitative criteria, necessary for determining the stability of an LC, are developed. In Section IV, a stability analysis of LCs is presented. The analysis provides insight into how long it takes before a perturbed SDM steps out of a LC. In Section V, the concepts of the foregoing sections will be used to obtain numerical results. Finally, in Section VI, conclusions will be presented.

II. Mathematical Background

A. State-Space Description

Though state-space descriptions of discrete-time processes are well established [13], in this section certain aspects are reviewed in order to present the paper in a self-contained way. The state-space description of the SDM in Fig. 2 will be examined as an illustration of a feedforward topology. This figure displays a feedforward SDM of order $N = 5$ with two resonator sections and the optional inclusion of dither just prior to quantization. This represents a typical modulator design, which is often used in practical designs [14]. The state-space description of other classes of modulators closely resembles the description that follows below [15]. One can easily read that, for the SDM depicted there

$$\begin{aligned} v^{(n)} &= \sum_{i=1}^{N} c_i s_i^{(n)} \\ y^{(n)} &= \text{sign}(v^{(n)}) \end{aligned} \tag{1}$$

where $y^{(n)}$ is the output bit at clock cycle n, $v^{(n)}$ is the quantizer input signal, and $s_i^{(n)}$ are the integrator outputs, called state variables. The c_i are the feedforward coefficients, and the f_i the resonator coefficients determining the positions of the poles in the loopfilter. The propagation of the states $\mathbf{s}$ can be written in matrix notation as

$$\begin{aligned} v^{(n)} &= \mathbf{c}^T \mathbf{s}^{(n)} \\ \mathbf{s}^{(n+1)} &= \mathbf{A}\mathbf{s}^{(n)} + (u^{(n)} - y^{(n)})\mathbf{d} \end{aligned} \tag{2}$$

where $\mathbf{A} \in \mathbb{R}^{N \times N}$ is called the *transition matrix* of the SDM of order N; $\mathbf{d} = (1, 0, \ldots, 0)^T \in \mathbb{R}^N$ describes how the input and feedback are distributed. The power of the state-space description is that it allows us to create a very compact description of the propagation of the SDM from time $t = 0$ to time $t = n$, as repeated application of (2) to $\mathbf{s}^{(0)}$ leads to $\mathbf{s}^{(n)}$

$$\mathbf{s}^{(n)} = \mathbf{A}^n \mathbf{s}^{(0)} + \left[\sum_{i=0}^{n-1} (u^{(i)} - y^{(i)}) \mathbf{A}^{n-i-1} \right] \mathbf{d}. \tag{3}$$

From the above equation, it is clear that all useful information is embedded in the second term of the right-hand-side (RHS) of (3); the first term carries no input signal information.

B. General Formulation of LC Conditions

In the introduction, the following definition of a LC was provided.

An LC is a sequence of P output bits, which repeats itself indefinitely.

In dynamical systems theory, an LC of period P exists if, for initial conditions $s^{(0)}$, P is the smallest positive integer such that

$$\mathbf{s}^{(P+n)} = \mathbf{s}^{(n)} \tag{4}$$

for all n greater than or equal to zero. The state-space variables then describe a periodic orbit in state space. However, from a practical point of view, an LC represents periodic behavior in the output y. Henceforth, we will refer to LC when periodic behavior in the output y is meant, and "periodic orbit in state space" when periodicity in the state variables is meant. In Appendix A, it is proven that, under reasonable assumptions, periodicity in y implies that a periodic orbit in state space exists. However, such equivalence is by no means always true. Thus, under the assumptions stated in Appendix A, there is a strict set of necessary (but not sufficient!) equalities that need to hold for the initial states if periodic output is to be sustained

$$(\mathbf{I} - \mathbf{A}^P)\mathbf{s}^{(0)} = \left[\sum_{i=0}^{P-1} (u^{(i)} - y^{(i)}) \mathbf{A}^{P-i-1} \right] \mathbf{d} \equiv \mathbf{L}_P(\{y^{(i)}\}) \tag{5}$$

where $\mathbf{L}_P(\{y^{(i)}\})$ has been introduced to avoid cumbersome notation. Formally

$$\mathbf{s}^{(0)} = (\mathbf{I} - \mathbf{A}^P)^{-1} \mathbf{L}_P(\{y^{(i)}\}). \tag{6}$$

From (5), one can obtain a unique value for the initial state $s^{(0)}$ if, and only if, the inverse of the matrix $(I - \mathbf{A}^P)$ exists. This will be elaborated in Section III; for now, it is assumed that a solution or solution space to (6) exists.

So far, the appearance of the LC has not been specified, except that it is of period P. If a LC is now defined as a specific sequence $\{y^{(i)}\}(i=1,\ldots,P-1)$, then for each $y^{(i)}$

$$y^{(i)}v^{(i)} = y^{(i)}\mathbf{c}^T\mathbf{s}^{(i)} > 0 \tag{7}$$

which is a test that has to be passed if a LC of the specified sequence $\{y^{(i)}\}(i=1,\ldots,P-1)$ exists. The inaccuracy made in (7) is that the possibility that $v^{(i)}y^{(i)}=0$ has been left out. As this equality occurs with probability zero over the continuously variable value of $v^{(i)}y^{(i)}$, this should not pose much of a problem. Thus, there is a set of equalities, (5), and a set of inequalities, (7), that need to be fullfilled in order to have a valid LC. Substitution of (3) in (7) gives

$$y^{(k)}\mathbf{c}^T\mathbf{A}^k\mathbf{s}^{(0)} + y^{(k)}\mathbf{c}^T\left[\sum_{i=0}^{k-1}(u^{(i)}-y^{(i)})\mathbf{A}^{k-i-1}\right]\mathbf{d} > 0, \quad k=0,\ldots,P-1 \tag{8}$$

which is equivalent to

$$y^{(k)}\mathbf{c}^T\mathbf{A}^k\mathbf{s}^{(0)} + y^{(k)}\mathbf{c}^T\mathbf{L}_k(\{y^{(i)}\}) > 0; \ \ k=0,\ldots,P-1 \tag{9}$$

by defining $[\sum_{i=0}^{k-1}(u^{(i)}-y^{(i)})\mathbf{A}^{k-i-1}]\mathbf{d} \equiv \mathbf{L}_k(\{y^{(i)}\})$.

Hence, one needs to simultaneously solve (5) and (9) in order to have a valid LC; in the next section more specific solutions will be derived for various SDM topologies.

III. Limit Cycle Conditions for Specific SDM Architectures

In order to quantify the importance of any disturbance of a LC, one first needs to solve (6). However, in the previous section, the remark has been made that the matrix $(\mathbf{I}-\mathbf{A}^P)$ may not be invertible. This observation carries significant practical relevance. The poles of the loop filter of an SDM are given by the eigenvalues of the transition matrix $\mathbf{A}$. Each pole p_i can be written as $p_i=\rho_i e^{j\omega_i}$, where ω_i is the pole frequency [4]. Hence, for a classical SDM which has all its loopfilter poles at dc, all eigenvalues of $\mathbf{A}$ will be one, as a result of which the inverse of (5) does not exist – hence, there is no unique solution to $\mathbf{s}^{(0)}$. On the other hand, if one has an SDM of even order N with $N/2$ resonator sections, all loopfilter poles will occur for frequencies other than dc. *As a result, there exists one and one only initial state* $\mathbf{s}^{(0)}$ *that results in a specific LC.* Most often, SDMs have at least a single zero at dc to avoid dc drift. In the following, we will distinguish between two main categories of SDMs. Those with and without poles at dc. The SDMs with poles at dc will be further subdivided in two categories. Those with poles at dc for the last two integrator sections; and those with poles away from dc for the last two integrator sections.

A special case of LC break up is due to *dithering* the SDM. Typically, dithering is achieved by adding a random number to the input of the quantizer, which therefore adds a random element to the quantization process. Because it is a special, but important case, and as its effectiveness is strongly related to the LC conditions, its discussion is included in Section III.C.

A. SDMs With DC Poles

In the case that the SDM has at least one of its poles at dc, the matrix $\mathbf{I}-\mathbf{A}^P$ is singular, and hence not invertible. To solve (5) for that case, one may use the singular-value decomposition (SVD) [16] of $(\mathbf{I}-\mathbf{A}^P)$

$$(\mathbf{I}-\mathbf{A}^P) = \mathbf{U\Sigma V}^T \tag{10}$$

where $\mathbf{\Sigma}\in\mathbb{R}^{N\times N}$ is a diagonal matrix whose elements σ_i are the singular values of $\mathbf{I}-\mathbf{A}^P$. The matrices $\mathbf{U}\in\mathbb{R}^{N\times N}$ and $\mathbf{V}\in\mathbb{R}^{N\times N}$ are the left and right singular vectors, respectively. Because both $\mathbf{U}$ and $\mathbf{V}$ are unitary, we also have $\mathbf{UU}^T=\mathbf{VV}^T=\mathbf{I}$. When the SDM is not reducible, exactly one of the singular values σ_i will be zero as a result of the fact that the loop filter displays a pole at dc. When the singular values are ordered in descending fashion, this singular value will be $\sigma_N=0$. This has the interesting consequence, that the last column of $\mathbf{V}$ is a *nonrelevant* direction, since it is always multiplied by $\sigma_N=0$. This last column of $\mathbf{V}$ will be denoted $\mathbf{v}_0$ (the so-called null space of $\mathbf{I}-\mathbf{A}^P$: $(\mathbf{I}-\mathbf{A}^P)\mathbf{v}_0=0$). Now, if a single solution is known (say, $\mathbf{s}_{\mathrm{mn}}$) to (5), any solution $s^{(0)}$ can be expressed as

$$\mathbf{s}^{(0)} = \mathbf{s}_{\mathrm{mn}} + \lambda\mathbf{v}_0. \tag{11}$$

In other words, the complete set of solutions to (5) is a *line*. Thus, for an N^{th} order SDM, at least $N-1$ (initial) conditions need to be fulfilled in order to have a LC.

In addition, the SVD is helpful in obtaining an initial solution to (5). Similar to (10),(11)

$$(\mathbf{I}-\mathbf{A}^P)^T = \mathbf{V\Sigma U}^T; \quad (\mathbf{I}-\mathbf{A}^P)^T\mathbf{u}_0 = 0 \tag{12}$$

where $\mathbf{u}_0$ is the null space of $(\mathbf{I}-\mathbf{A}^P)^T$. Therefore, (12) is equivalent to

$$\mathbf{u}_0^T(\mathbf{I}-\mathbf{A}^P) = 0. \tag{13}$$

Multiplying both sides of (5) with $\mathbf{u}_0^T$ one obtains

$$\mathbf{u}_0^T(\mathbf{I}-\mathbf{A}^P)\mathbf{s}^{(0)} = \mathbf{u}_0^T\mathbf{L}_P(\{y^{(i)}\}) = 0 \tag{14}$$

stating a necessary condition $\mathbf{u}_0^T\mathbf{L}_P(\{y^{(i)}\})=0$ for the existance of a solution to (5). For the type of SDMs that are investigated in this section, with a pole at dc (and thus infinite gain for dc) this condition is equivalent to the intuitively obvious condition that the average input should equal the average output of the SDM

$$\frac{1}{P}\sum_{i=0}^{P-1}u^{(i)} = \frac{1}{P}\sum_{i=0}^{P-1}y^{(i)}. \tag{15}$$

When the SDM input is a constant dc value, $u^{(i)}\equiv u$ and the sequence $\{y^{(i)}\}$ also completely determines the input u to the SDM.

Second, if a solution to (5) exists, one can define $\mathbf{s}_{\rm mn}$ as a minimum norm solution to (5) [16]

$$\mathbf{s}_{\rm mn} = \mathbf{V}\mathbf{\Sigma}'\mathbf{U}^T\mathbf{L}_P(\{y^{(i)}\}); \quad \mathbf{\Sigma}' = {\rm diag}\left(\frac{1}{\sigma_1},\frac{1}{\sigma_2},\ldots,\frac{1}{\sigma_{N-1}},0\right). \tag{16}$$

The solution $\mathbf{s}_{\rm mn}$ is characterized by the fact that the norm $|\mathbf{s}_{\rm mn}|$ is the least of all norms $|\mathbf{s}^{(0)}|$ of other solutions to $\mathbf{s}^{(0)}$.

Though one part of the necessary conditions for a LC has been solved (5), the set of inequalities represented in (9) still needs to be solved. For each inequality k in (9), one can write an equality which represents the conditions under which the constraint is on the edge of being violated

$$y^{(k)}\mathbf{c}^T\mathbf{A}^k\mathbf{s}^{(0)} + y^{(k)}\mathbf{c}^T\mathbf{L}_k(\{y^{(i)}\}) = 0; \;\; k = 0,\ldots,P-1. \tag{17}$$

This represents an $N-1$-dimensional hyperplane which bisects the N dimensional space. The point where this surface intersects the line defined by (11) represents the boundary where a LC of length P is on the verge of violating the $k^{\rm th}$ constraint. This point is given by solving for $\lambda_k^{\rm crit}$ in the equation

$$y^{(k)}\mathbf{c}^T\mathbf{A}^k(\mathbf{s}_{\rm mn} + \lambda_k^{\rm crit}\mathbf{v}_0) + y^{(k)}\mathbf{c}^T\mathbf{L}_k(\{y^{(i)}\}) = 0. \tag{18}$$

Equation (18) defines a distance $\lambda_k^{\rm crit}$ from the initial point $\mathbf{s}_{\rm mn}$ at which the $k^{\rm th}$ constraint is on the edge of being violated. Depending on the sign of $y_k\mathbf{c}^T\mathbf{A}^k\mathbf{v}_0$, either $\lambda > \lambda_k^{\rm crit}$ (sign positive) or $\lambda < \lambda_k^{\rm crit}$ is required in order to fulfill the $k^{\rm th}$ constraint. The set of constraints, (18), can be divided into two categories, $\lambda_>^k$ and $\lambda_<^k$, of feasible λ, depending on the sign of $y_k\mathbf{c}^T\mathbf{A}^k\mathbf{v}_0$

$$\forall k = 0,\ldots,P-1:$$

$$\text{if } y^{(k)}\mathbf{c}^T\mathbf{A}^k\mathbf{v}_0 > 0, \text{ then } \lambda > \lambda_>^k = \lambda_k^{\rm crit} \tag{19}$$

$$\text{if } y^{(k)}\mathbf{c}^T\mathbf{A}^k\mathbf{v}_0 < 0, \text{ then } \lambda < \lambda_<^k = \lambda_k^{\rm crit}. \tag{20}$$

Defining

$$\lambda_> = \max_k(\lambda_>^k) \tag{21}$$

$$\lambda_< = \min_k(\lambda_<^k) \tag{22}$$

provides an interval $[\lambda_>, \lambda_<]$ for a feasible λ

$$\lambda_{\rm feas} \in [\lambda_>, \lambda_<]. \tag{23}$$

Obviously, when $\lambda_> > \lambda_<$, there is no feasible solution, and the LC $\{y^{(i)}\}$ cannot exist.

We will now investigate the nature of the disturbance that can be applied to the SDM, before the LC breaks up. We will separate the two situations for SDMs with dc poles: 1) SDMs with (more than) one dc pole, where the last integrator section creates a pole at dc (in Fig. 2 this corresponds with $f_2 = 0$); and 2) SDMs with (more than) one dc pole where the last two integrator sections create a resonator with poles away from dc (in Fig. 2 this corresponds with $f_2 \neq 0$).

1) Last Integrators With DC Poles: The question that needs to be answered, is what the null space in (11) looks like. For the current case, the last two integrator sections create two dc poles, which translates to the fact that the last column of the transition matrix $\mathbf{A}$ is given by $\mathbf{A}(:N) = (0,\ldots,0,1)^T$. The ith element of the last column N of $\mathbf{A}^2$ is given by

$$(\mathbf{A}^2)_{iN} = \sum_k^N \mathbf{A}_{ik}\mathbf{A}_{kN}. \tag{24}$$

Because of the special structure of $\mathbf{A}$, we know that $\mathbf{A}_{kN} = \delta_{kN}$, where δ is the Kronecker delta. Hence, we have

$$(\mathbf{A}^2)_{iN} = \sum_k^N \mathbf{A}_{ik}\delta_{kN} = \mathbf{A}_{iN} = \delta_{iN} \tag{25}$$

which again is a matrix with the last column $(0,\ldots,0,1)^T$. By induction, $\mathbf{A}^P$ inherits this special structure, too.

Therefore, the last column of $(\mathbf{I}-\mathbf{A}^P)$ equals $(0,\ldots,0)^T$, and, hence, the null space of $(\mathbf{I}-\mathbf{A}^P)$ contains at least $(0,\ldots,0,1)^T$. Because for these $N^{\rm th}$ order SDMs the rank of $\mathbf{A}$ equals $N-1$, the null space is $(0,\ldots,0,1)^T$.

Referring to Fig. 2 to see the implication of this, it means that the state of the last integrator can be altered over a range $[\lambda_>, \lambda_<]$, without breaking up the LC. However, the effect of changing the last integrator state is nothing other than adding an offset just before the quantizer. For example, when the last integrator state is changed by an amount δ, this is equivalent to adding a value ν to the input of the quantizer with $\nu = c_N\delta$. Hence, this approach provides the means to define a minimum disturbance, just before the quantizer, which is necessary to break up a LC in an SDM with a dc pole for the last integrator.

2) Last Integrators With Poles Away From DC: In case the last two integrator sections form a resonator, the last column of $\mathbf{A}$ will be of the form $\mathbf{A}(:N) = (0,\ldots,0,f,1)^T$, and, clearly, the null space does not have the simple shape anymore as in the previous section. In fact, if the feedback coefficient in the last two integrator sections equals f, it can be shown that to very good approximation the null-space is given by

$$\mathbf{v}_0 = (0,\ldots,f,0,1)^T. \tag{26}$$

Hence, in order not to disturb a LC when changing the last integrator section, the third integrator state should also be changed.

Although it is not as easy to determine the exact minimum disturbance that needs to be applied just before the quantizer (i.e., change the last integrator state) before the output bit changes sign, it is still possible to define a disturbance which is equal or larger than this minimum amount. One can do this under the assumption that $f \ll 1$, in which case one can apply the theory from the previous section. As typical values for f are of the order of 10^{-3} (see Appendix C), validity of the approximation is asserted. This approximation slightly overestimates the minimum disturbance required to break up a LC, and thus represents a pessimistic estimate.

B. No DC Poles

A special situation arises when the SDM has no dc poles. In that case, the null space of $(\mathbf{I}-\mathbf{A}^P)$ is zero. There is only one solution $\mathbf{s}^{(0)}$ to (5). If this solution also complies with all inequalities (9), it results in a LC. Because the null space is zero, any change of the integrator states would result in a break-up of the LC. A relevant question that remains, however, is how long it would take before the bit-pattern is changed from the

LC pattern; in other words, what freedom does one have when the only requirement is to fulfill (9). This will be the subject of Section IV. Note, that the system of inequalities itself would lead to the same solution as (5) would after an infinite amount of time (see Appendix A).

C. Dither

The basic principle of "dithering" (adding random offsets to the quantizer) is sketched in Fig. 2. The addition of dither represents a special case of LC disturbance, since it does not directly influence the integrator values. The only way in which dither can break up a LC is by changing the sign of the input to the quantizer, causing it to create a bit-flip in the LC output. The minimum amplitude $|\nu_{\min}|$ of the dither that is necessary to certainly break up a LC, is easily determined as

$$\nu_{\min} = \min_i |v^{(i)}(\mathbf{s}^{(0)})|, \qquad i = 0, \ldots, P-1 \tag{27}$$

where the dependence of the minimum dither level on the initial states is explicitly indicated. In a typical situation, where dither according to a certain (*e.g.*, rectangular) pdf spanning a width W is applied, all dither values with amplitude less than $\nu_{\min}(\mathbf{s}^{(0)})$ are without any effect. Because of its dependence on the initial states of the SDM, (27) is not the most convenient expression to determine an appropriate dither level. Preferably, one would have the expression that provides the maximum of $\nu_{\min}$ over the initial states. In Section III-A1), it was derived that for SDMs with the last integrators having their poles at dc, the value of the last integrator could vary over a range $[\lambda_>, \lambda_<]$ without breaking up the LC. The interpretation is that

$$\text{if } \lambda = \lambda_>, \lambda_< \text{ then } \min_i |v^{(i)}| = 0, \qquad i = 0, \ldots, P-1. \tag{28}$$

Because the quantizer input v bears a linear relation to λ, it means that the minimum amplitude dither $\nu_{\min}$, needed to break up a LC, is maximized over all $\mathbf{s}^{(0)}$ when $\lambda = (\lambda_> + \lambda_<)/2$, and thus

$$\nu_{\min} = \mathbf{cv}_0 \frac{\lambda_< - \lambda_>}{2}. \tag{29}$$

For most SDMs, the value $\nu_{\min}$ can be easily determined using results obtained previously, without resorting to (27). For the SDMs which have a resonator section as last integrators, but also have dc poles, as discussed in Section III-A2), the null-space is not exactly equivalent to a mere change of the last integrator value. However, for typical SDMs, the value of the feedback coefficient of the last resonator is much less than 1, and therefore the null space is *almost* equivalent to a change of the last integrator state. As a result, one can treat such SDMs in exactly the same way for determining a lower bound to the minimum amount of dither.

For SDMs without dc poles, however, the null-space has dimension zero and the methods outlined above cannot be used anymore. In this case, the only option is to determine the minimum amount of dither through (27).

IV. STABILITY ANALYSIS OF LIMIT CYCLES

To determine whether a LC is stable, the same assumption made in Section III.C will be made that any disturbance that causes a bit-flip with respect to the ideal LC pattern, causes break-up of the LC. An approach based on perturbation theory will be followed in order to determine when such a bit-flip will occur.

For a given LC of length P, the states at clock cycle P are given by

$$\mathbf{s}^{(P)} = \mathbf{A}^P \mathbf{s}^{(0)}. \tag{30}$$

To have some idea about stability of LCs, the original state variable $\mathbf{s}^{(0)}$ will be perturbed by an amount $\boldsymbol{\epsilon}^{(0)}$

$$\hat{\mathbf{s}}^{(0)} = \mathbf{s}^{(0)} + \boldsymbol{\epsilon}^{(0)}. \tag{31}$$

The growth of a disturbance $\boldsymbol{\epsilon}$ in the state variables after M periods (and, hence $n = MP$ clock cycles) of the LC is given by

$$\hat{\mathbf{s}}^{(MP)} = \mathbf{A}^{MP}(\mathbf{s}^{(0)} + \boldsymbol{\epsilon}^{(0)}) = \mathbf{s}^{(0)} + \mathbf{A}^{MP}\boldsymbol{\epsilon}^{(0)} = \mathbf{s}^{(0)} + \boldsymbol{\epsilon}^{(MP)}. \tag{32}$$

To analyze (32), a Jordan decomposition [16] of $\mathbf{A}$ is created, which is defined as[1]

$$\mathbf{A} = \mathbf{VJV}^{-1} \tag{33}$$

where $\mathbf{J}$ is a Jordan matrix of the form

$$J_{ij} = \begin{cases} \mu_i, & \text{if } i = j \\ 0, 1, & \text{if } j = i - 1 \\ 0, & \text{otherwise} \end{cases} \tag{34}$$

with μ_i the ith eigenvalue of the transition matrix $\mathbf{A}$. The main advantage of this decomposition is that it provides a compact representation of repeated application of $\mathbf{A}$ as

$$\mathbf{A}^k = \mathbf{VJ}^k\mathbf{V}^{-1} \tag{35}$$

where $J_{ii}^k = \mu_i^k$. From this expression, it is evident that for SDMs with eigenvalue magnitudes $|\mu_i| > 1$, multiple application of $\mathbf{A}$ will result in exponential growth of the disturbance. When $|\mu_i| < 1$, on the other hand, exponential decay will occur. The effect of a disturbance will be studied in the next sections, both for SDMs with all poles at dc (all eigenvalues $|\mu_i| = 1$), and for SDMs with resonator sections and eigenvalues $|\mu_i| \neq 1$.

A. Only DC Poles

In the special case where all the poles of the loop filter are at dc, all eigenvalues $|\mu| = 1$, as a result of which only polynomial growth can occur. In particular, when the eigenvalues are all unity, the result for $\mathbf{J}^n$ can be written as

$$J_{ij}^k = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i < j \\ \begin{pmatrix} k \\ k - (i-j) \end{pmatrix}, & \text{otherwise.} \end{cases} \tag{36}$$

For an SDM with only dc poles, the transition matrix is exactly in the shape of this Jordan block, with eigenvalues equal to 1, and no further decomposition is necessary (SDMs which exhibit poles in the loopfilter, are not in Jordan form). In order to determine when a LC will be broken up, the disturbance $\delta v^{(MP)}$

[1]Alternatively, the Schur decomposition could be used, which, for the types of SDMs under consideration, is less practical.

at the quantizer input that results in a bit-flip must be determined, akin to the discussion in Section III-C. From (2)

$$\delta v^{(MP)} = \mathbf{c}^T \mathbf{A}^{(MP)} \boldsymbol{\epsilon}. \tag{37}$$

Using (36), and making the approxiation that $MP \gg 1$, the polynomial divergence of a modulator of order N can be approximated by

$$\begin{aligned}\delta v^{(MP)} &\approx \epsilon_1 \left(\frac{P^{N-1}}{(N-1)!} c_N M^{N-1} + \frac{P^{N-2}}{(N-2)!} c_{N-1} M^{N-2} + \cdots \right) \\ &+ \epsilon_2 \left(\frac{P^{N-2}}{(N-2)!} c_{N-1} M^{N-2} + \frac{P^{N-3}}{(N-3)!} c_{N-2} M^{N-3} + \cdots \right) \\ &+ \vdots \\ &\epsilon_N c_N \end{aligned} \tag{38}$$

where $\mathbf{c} = (c_1, \ldots, c_N)^T$ and likewise for $\boldsymbol{\epsilon}$. Hence, if the first integrator is disturbed, the number of LC periods M_c it takes before a LC is broken up can be approximated by

$$M_c \approx \frac{1}{M} \left(\frac{(N-1)! |\delta v_{\text{crit}}|}{\epsilon_1 c_N} \right)^{1/(N-1)} \tag{39}$$

where δv_{crit} is the critical value of δv where the LC is broken up. From Section III-A, we have that

$$\delta v_{\text{crit}} \in [-\mathbf{c}\mathbf{v}_0(\lambda_< - \lambda_>), \mathbf{c}\mathbf{v}_0(\lambda_< - \lambda_>)] \tag{40}$$

where the precise value of δv_{crit} depends on the initial state $\mathbf{s}^{(0)}$.

B. Resonator Sections

In case of resonator sections, it proves to be slightly more difficult to obtain a general algebraic expression for the rate of growth of $|\delta v|$. An expression for the effect of a disturbance can be obtained from the difference equations describing a resonator section, though. From (73) in Appendix B, it is shown that, for small f, the growth of a disturbance in a single resonator section can be described by

$$y_r^{(n)} \approx \frac{u_r^{(0)}}{f} \cos(n \arctan(\sqrt{f}))(\sqrt{1+f})^n + \frac{u_r^{(n)}}{f}, \qquad n \geq 0 \tag{41}$$

where $u_r^{(i)}$ is the input at clock cycle i to the resonator, and $y_r^{(i)}$ is the output of the resonator (or, effectively, the output of the last integrator of the resonator) at clock cycle i; f is the feedback coefficient in the resonator section (see also Fig. 2). The time instant $n = 0$ is (a bit arbitrarily) taken as a moment where the bit output pattern coincided with the bit output pattern corresponding to the LC. It can be observed that the original input to the resonator, $u_r^{(n)}$, appears at the output, multiplied by $1/f$, in addition to a sine or cosine, which is of the same order of magnitude as $u_r^{(n)}/f$. The output of a series of resonators, each characterized by a feedback coefficient f_i, will thus consist of a superposition of sinusoidal signals, with frequency $\arctan(\sqrt{f})$, each of them exponentially diverging as $e^{n \ln(1+\sqrt{f})}$. The input to the first resonator will appear at the output of the last resonator, multiplied by a factor $1/(f_1 \ldots f_p)$ if p resonator sections are cascaded. From $n = 0$, the output of the series of resonators will continue to increase steeply, until the maximum of the sine wave with the lowest frequency is reached. This maximum occurs when

$$\cos(n \arctan(\sqrt{f})) \approx -1; \quad n \approx \frac{\pi}{\sqrt{f}}; \rightarrow M \approx \frac{\pi}{P\sqrt{f}}. \tag{42}$$

For practical f of about 10^{-3}, this means that n is of the order of 100 when this maximum is achieved. The exponential factor has typically grown a negligable 20% by then, which shows that when δv_{crit} has not been reached, LC break up can only be achieved through the exponential growth. Only for $n \approx 10^4$, the rate of growth of the exponential divergence equals that of the polynomial growth.

If an SDM consists of a cascade of simple integrators and resonators, the output will be a multiplication of the polynomial divergence as described in Section IV-A, and the oscillatory behavior as described above.

C. Implications

From (38) and (42), two important observations can be made. Firstly, because in both cases $\delta v^{(MP)}$ will be dominated by the highest order terms, it will be the weighting coefficient c_N of the last integrator that determines the rate at which $\delta v^{(MP)}$ grows. Hence, more aggressive noise shapers, which have relatively large weight at the high order integrators, will exhibit much less stability with respect to LCs as mild noise shapers will. Secondly, the growth rate of $\delta v^{(MP)}$ is in both cases increasing with P, which means that long LCs are much less stable, and hence, much less likely to occur, than short LCs.

Finally, it is worth mentioning that disturbing the SDM just before the quantizer (dithering the quantizer), is apparently less effective than disturbing the SDM at any other position. Placing the disturbance anywhere else guarantees that the LC will always be broken up, no matter how small the disturbance is.

V. Numerical Results

The results of the work detailed in the preceding part, have been used to obtain some results on several different noise transfer functions [(NTFs), all Butterworth design], which have been implemented in feedforward SDMs. The SDMs, all with an oversampling ratio of 64, have been chosen to illustrate the difference in behavior for various SDMs with aggressive noise shaping and mild noise shaping. In the case of most aggressive noise shaping, the NTF has a high corner frequency of $f_c = 160$ kHz. Although this SDM displays excellent noise suppression in the baseband, its stability is severely compromised due to which it is of hardly any practical use. The SDM with mild noise shaping has its NTF with corner frequency at 80 kHz. The naming convention is such, that the first part of the name of the SDM reflects its NTF corner frequency, and the last part is either "a" (meaning with resonator sections, i.e., $f_1 \neq 0$ and $f_2 \neq 0$) or "b" (meaning without resonator sections, i.e., $f_1 = f_2 = 0$). A more detailed description of all the SDMs used in this paper, along with full descriptions of the SDMs with NTF corner frequencies at 120 and 80 kHz are given in Appendix C.

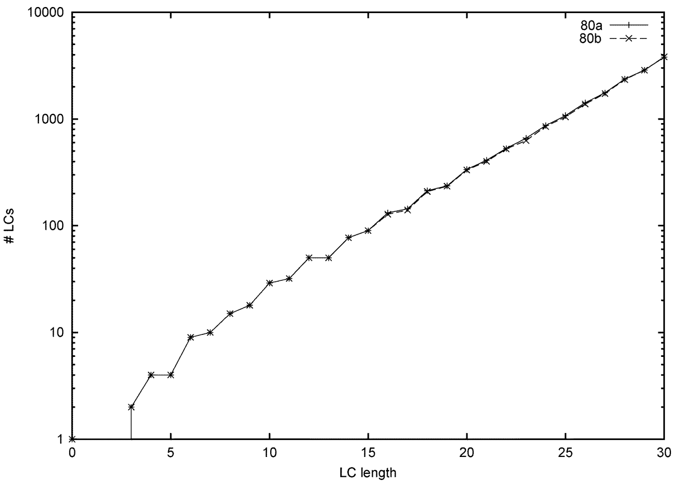

Fig. 3. Occurrence of LCs of SDM 80a and 80b.

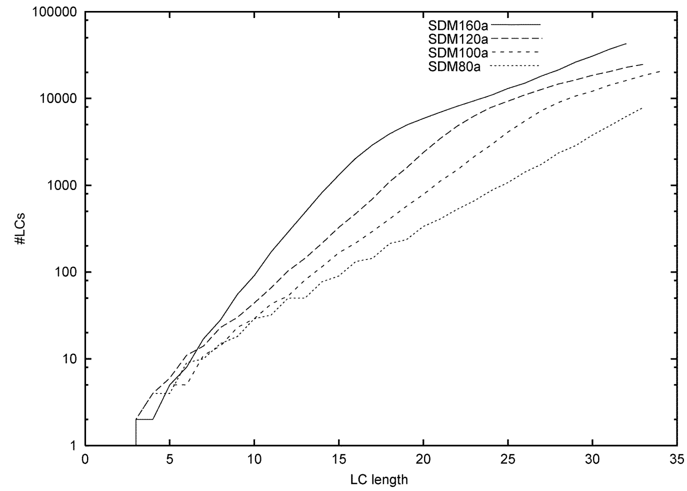

Fig. 4. Occurrence of LCs of SDMs 80a, 100a, 120a, and 160a.

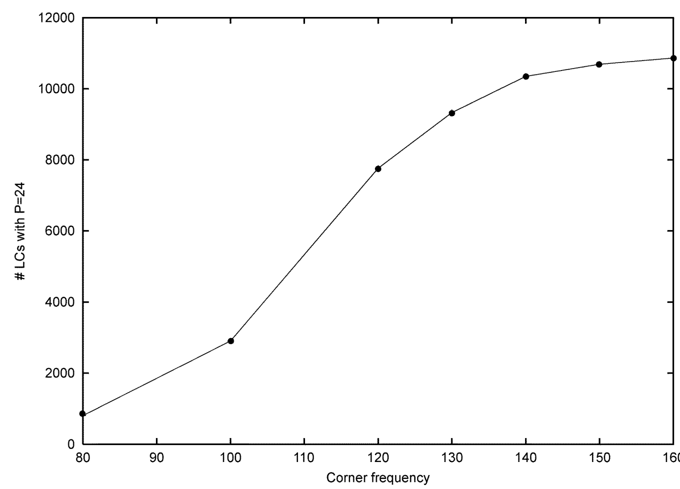

Fig. 5. Number of LCs for a fixed *LC* period of 24, as a function of the NTF corner frequency used in the SDM design.

In the following sections, we will discuss results on static and dynamic behavior of SDMs. Feedforward topologies are presented to judge how the implementation topology influences the LC behavior.

A. Static Behavior

In Fig. 3, the occurrence of LCs for SDMs with and without resonators is presented as a function of LC length. This occurrence has been obtained by generating all independent bit patterns for a given length, and checking whether this pattern could represent a specific LC with the theory presented in Section III.

Fig. 3 represents all independent LCs that could exist with lengths ranging from 3 to 30 bits. All possible dc values are represented by these LCs, and though some of these LCs theoretically exist, they cannot occur in practice. For example, it is possible to define an LC corresponding to an input of 0.9, where none of the SDMs studied would be capable of representing this dc level without running into instability. It is apparent immediately from Fig. 3, that the presence or absence of resonator coefficients for the SDM with NTF corner frequency at 80 kHz is immaterial to the number of LCs that can occur. The same is approximately true when comparing other SDMs with and without resonator section. However, a significant difference is displayed when comparing the aggressive SDM 120a and the nonaggressive SDM 80a in Fig. 4. Rather counterintuitive, the SDM 120a displays *more* LCs than SDM 80a; one would expect the reverse to be true, as experimental evidence usually proves stable SDMs more susceptible to LCs than aggressive SDMs. The SDMs all show an initial steep growth of the number of LCs, followed by a transition to a region of less steep growth. Based on pure permutations, one would expect the number of LCs to grow proportionally to 2^P. While, indeed, exponential growth of the number of LCs is observed, from a numerical fit the initial growth for SDM 120a is proportional to $2^{0.55P}$, and for SDM 80a $2^{0.36P}$. Above the cross-over point, the growth is proportional to $2^{0.17P}$, and this is approximately true up to the largest LC investigated for all SDMs independent of their aggressiveness. Also, the frequency of the cross-over point appears to be coincident with the LC period that corresponds to the corner frequency of the Butterworth high-pass filter that was used in the design. To further illustrate this behavior, the dependence of the number of LCs at given LC length ($P = 24$) on the corner frequency of the Butterworth NTF design is given in Fig. 5. This clearly illustrates the increase of the number of LCs with increased aggressiveness of the SDM, and also shows that for highly aggressive SDMs the number of possible LCs is virtually constant.

Qualitatively, these observations can be explained on the basis of the phase characteristic of the loop filter. To sustain a LC perfectly, the phase shift for all frequency components of the LC needs to be π (the feedback loop accounts for another factor π, summing up to the required 2π corresponding to a delay of one period). For high frequencies, down to the corner frequency of the loopfilter, its phase shift deviates relatively little from π. Below that frequency, it starts to deviate strongly from π, corresponding to the fact that the likelihood of a long LC with significant low frequency content being sustained is low. This causes the rate of growth of the number of LCs to reduce above the loopfilter corner frequency, *cf.* Fig. 4. Also, the more aggressive the loopfilter, the less deviation from π for high frequencies. Hence, an aggressive loopfilter will sustain more LCs with relatively large high-frequency content than a nonaggressive filter, *cf.* Fig. 5.

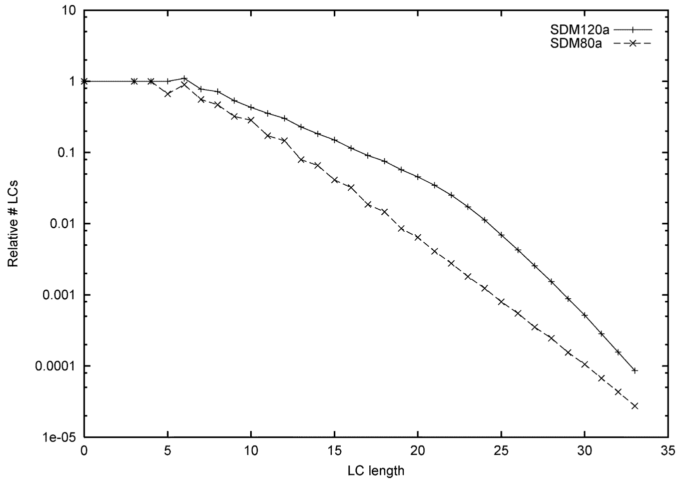

Fig. 6. Relative occurrence of LCs of SDM 120a and 80a with respect to all permutations of bits.

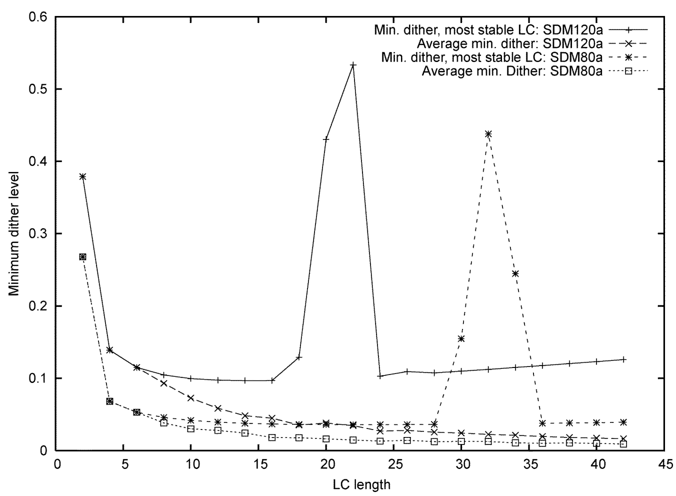

Fig. 7. Minimum level of dither needed to break up a LC corresponding to a dc input 0 .

While the absolute number of LCs increases rapidly, relative to the number of possible permutations of $+1$s and -1s it is reducing rapidly as is demonstrated in Fig. 6. The total number N_P of permutations for an LC of length P is approximately given by [3]

$$N_P \approx \frac{2^P}{P}. \tag{43}$$

The division by P corrects for the fact that of all 2^P permutations, exactly P represent a cyclicly shifted version of the same basic LC. To obtain the exact number of irreducible LCs, correction should be made too for the number of LCs that are a concatenation of smaller LCs. However, for reasonable P, this number is much smaller compared to N_P and thus ignored. In Fig. 7, the minimum dither level that is needed to certainly break up the most stable LC is depicted. In plusses ("+") and stars ("*"), the most stable LC for SDM 120a and 80a, respectively, for dc input is depicted. While slightly more stable LCs can sometimes be found for non-dc inputs, this does not represent a practical situation. The first interesting observation is that the LCs for the aggressive SDM 120a are *more stable* against dither

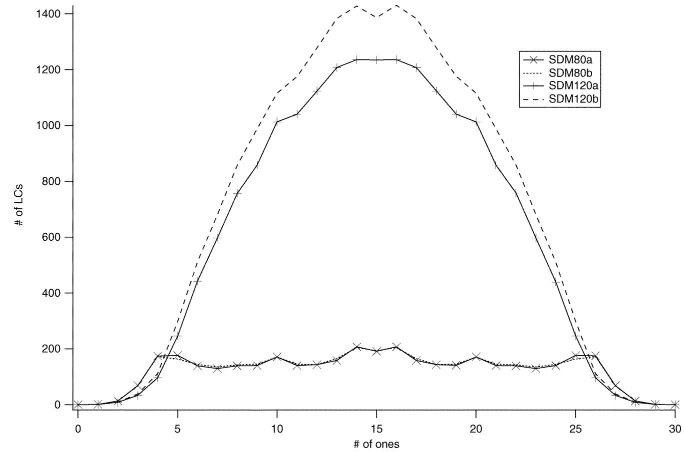

Fig. 8. Occurrence of LCs as a function of the number of 1's in the LC, for various SDMs. Zero dc level corresponds to 15 1's, 20 1's corresponds to a dc level of one third, *etc.*.

than those of the less aggressive SMD 80a. Again, this is quite counter-intuitive as we expect aggressive SDMs to be less susceptible to LCs. Also, we can see that there is a very stable LC occurring around LC length 22 for SDM 120a, and for LC length 32 for SDM 80a. Upon investigation of these LCs, it appeared that they consist of a series of 11 1s followed by 11 -1s for SDM 120a, and likewise 16 1s and 16 -1s for SDM 80a. This corresponds to a square wave of frequency 120 kHz and 80 kHz, respectively, which are exactly the corner frequencies of the NTF design of the SDMs. Although not shown, identical behavior occurs for other SDMs. In practice, however, these LCs require huge initial integrator states that could never occur. Long before such an integrator state could be reached in real operation, the SDM would have reached a state with unbounded state variables where the output bit pattern would not reflect the input signal anymore (see also [1] for a discussion on this phenomenon). As a result, if the SDM has been forced into this LC, the SDM runs unstable upon the slightest disturbance of the integrators.

This is to be contrasted with the LC behavior for other LC lengths. The shortest LC, the sequence $\{1, -1\}$, appears to be by far the most stable (disregarding the previously discussed LCs) for both SDMs. For longer LCs, the amount of dither needed for break-up decreases to a minimum value close to the peak, after which the LC becomes more stable. All these LCs consist of the sequence $\{-1, 1, -1, 1, \ldots, -1, 1, -1, -1, 1, 1\}$, which represents the minimally possible deviation for the simple $\{-1, 1\}$ sequence. While these most stable LCs slightly increase in stability for longer LCs, on average the amount of dither necessary for break-up decreases. This is indicated in crosses and squares for SDM 120a and 120b, respectively, in Fig. 7. The average amount of dither is defined as the average of the minimum dither levels that are needed to break up the individual LCs. Again, we see that SDM 120a presents LCs that are in general more stable than those of SDM 80a. At LC lengths of 42, the average amount of dither is reduced to about 0.03 and 0.017 for SDM 120a and 80a, respectively, which is consistent with the intuition that longer LCs represent more boundary conditions to be fulfilled and are thus more easy to break up. Another interesting characteristic to study is the relative preference of the SDM for LCs of a certain dc level. These results are displayed in Fig. 8.

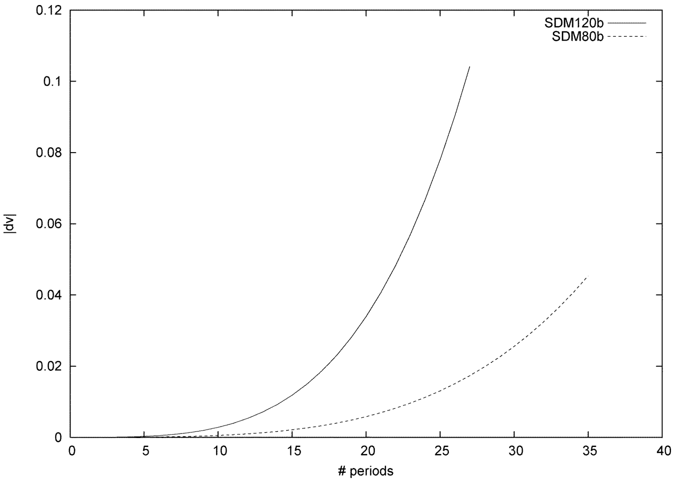

Fig. 9. Disturbance of identical LCs for SDM 120b and 80b (with all eigenvalues equal to 1) due to a small disturbance (-120 dB) on the first integrator state. Depicted is the quantizer input until the LC breaks up.

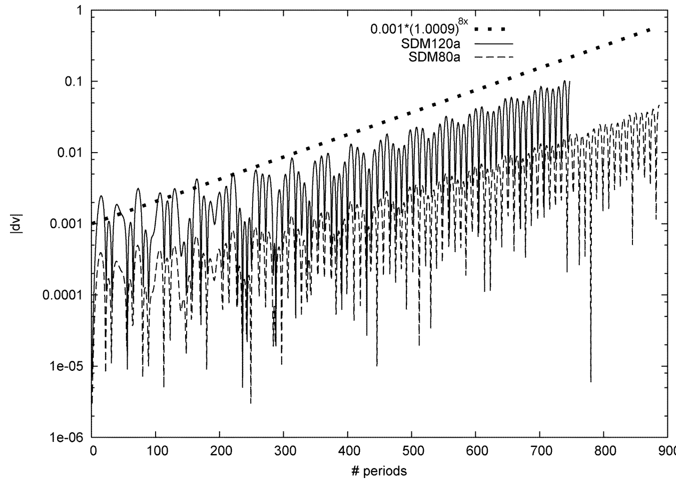

Fig. 10. Disturbance of LCs of length 10,20 and 30 for SDM 80a due to a small disturbance (-120 dB) on the integrator states. Depicted is the quantizer input until the LC breaks up. The dotted line represents the exponential growth that is expected due to the (largest) eigenvalues of the transition matrix.

Confirming the results in Fig. 4, in general SDM 120 exhibits many more possible LCs for any dc level than SDM 80. Also, we see that (as anticipated) the number of LCs is identical for a certain dc level and the negative dc level. However, whereas SDM 120 has strong preference for LCs with small absolute dc level, SDM 80 apparently has little preference! Moreover, where SDM 80 displays little dependence on the presence of resonator sections, SDM 120 shows, especially for the smallest dc levels some dependence, displaying most LCs when no resonators are present. The reason for this behavior is unclear.

B. Dynamic Behavior

In Fig. 9, the effect of a small disturbance of the integrator states on a LC is illustrated. It depicts the growth of $|\delta v|$, which is the deviation of the quantizer input from its ideal input, as defined in (37). The LC studied was the most stable of length 8, *i.e.*, $-1, -1, +1, +1$ followed by a sequence of 2 $-1, +1$ pairs. A disturbance of -120 dB (10^{-6}) was applied to the first integrator at time instant $n = 0$ in order to break up the LC. The effect of such a disturbance on the output signal during normal SDM operation is very small; in fact, it is much less than the effect that sufficiently dithering the quantizer would have had. Studying the case where all transition matrix eigenvalues equal 1, we can clearly see that for SDM 120b, which is the more aggressive one, the deviation $|\delta v|$ from the ideal quantizer input increases steeply. This, in turn, results in early break-up of the LC. For the nonagressive SDM 80b, this increase is much less steep. As both SDMs have all loop filter poles at the unit circle, the rate of growth of δv is polynomial. Even though the maximum deviation $|\delta v|$, for which LC break-up occurs for this SDM, is much less than for SDM 120b, it takes a 30% longer time for it to break up compared to the same LC in SDM 80b. Note, that the fact that the maximum deviation for which break-up occurs is larger for the least aggressive SDM, is in correspondance with the results on dither as presented in Fig. 7. Thus, aggressive noise shapers need *more* dither in order to break up a LC when this dither is added to the quantizer, and need *less* dither when the dither is added to the input of the SDM to break up a LC. Upon substitution of the relevant parameters in (39), break-up of the LC is predicted at the 38th period for SDM 80b, and at the 29th period for SDM 120b, in good agreement with the observed behavior. The same trend is observed in Fig. 10 for the SDMs 120a and 80a, which both display eigenvalues > 1; the largest eigenvalue, corresponding to the pole at 20 kHz, is $1.0000 \pm i0.042\,43$, which has a norm equal to 1.000 90. The dotted line in Fig. 10 represents the exponential growth that is expected on basis of this eigenvalue. As a result, δv displays both oscillatory and exponential growth as can be inferred from Fig. 9. However, a drastically larger number of LC periods passes, before it is broken up, in line with the expectations mentioned in Section IV. The first maximum in δv occurs, both for SDM 80a and 120a, at the 15th period, exactly equal to the prediction of (42). The predicted amplitude at these maxima equals 0.000 25 and 0.0017 for SDM 80a and 120a, respectively, which is about a factor of 1.5 too low with respect to the simulation result. This difference is easily explained because of the fact that the input to the cascade of resonators is a constant (due to the action of the first integrator). This means that the last term in (41) is nonzero, which effect is ignored in determining the maximum value. Because the first maximum is still far from δv_{crit}, the total duration of the LC is determined by the exponential growth. The predicted duration of the LC is 807 periods for SDM 80a, and 714 periods for SDM 120a, which is both about 5% shorter than observed in simulation.

VI. Conclusion

This work is an attempt to construct a general theory describing LCs in 1-bit SDMs, and to provide the designer with tools other than numerous simulations to obtain an insight into typical LC behavior of SDMs. It has been proven that, under almost all circumstances, LC behavior is observed in the output if and only if a LC occurs in state space. It has been shown that LC behavior can occur in a wide variety of situations.

In Section III.A, a recipe was given whereby, for constant input, *all* LCs of a given period can be found for any SDM with at least one pole at dc. Equation (16) provides a least squares solution to the LC conditions. If the constraint equations, (9) and (15), are also satisfied, then this is an exact solution. Equation (18) may then be solved to find the exact set of initial conditions, (11), which give rise to this LC. This same procedure can be applied when the SDM has no dc poles. However, it becomes simpler in this situation since (5) can now be solved directly, and (9) is the only constraint. The essential difference between these two situations is that, if constraint equations are satisfied, SDMs with dc poles will exhibit a line of initial conditions which give rise to a LC, whereas SDMs without dc poles will exhibit a unique solution. One immediate consequence of the initial condition dependence is that, for an SDM of order N with dc poles, $N-1$ states need to have a well-defined value, and all N states need to have a well-defined value for SDMs without dc poles. This makes LCs for higher order SDMs (which typically also exhibit more aggressive noise shaping) less likely to occur, especially when they do not exhibit poles at dc.

It is postulated that the most stable LC is that which is a sequence of $-1, +1$ pairs, followed by a single $-1, -1, +1, +1$ quartet. It has also been shown, that the classical approach of dithering the quantizer may not be the optimal way of removing LCs. Adding a small disturbance to an integrator state is far more efficient and will always result in break up of the LC. The noise penalty is rather limited, as the input disturbance can be made as small as 10^{-6} or -120 dB. Furthermore, it has been shown that very small changes in an SDMs structure can have significant effects on the rate of growth of any disturbance to a LC. SDMs with only dc poles will exhibit polynomial growth, whereas the inclusion of resonator sections or other modifications to the structure may yield exponential growth. However, if these modifications result in the transition matrix having complex conjugate pair eigenvalues, then the exponential growth is exhibited as the disturbance *spiraling* away from initial conditions. Thus, this exponential growth may take significantly longer to break up the LC than the polynomial growth which occurs without resonators. Therefore, in general, SDMs without resonators are less susceptible to LCs.

An important characterization given the goals of SDM design, is distinguishing LC behavior for SDMs with different noise shaping characteristics. Intriguingly, SDMs with aggressive noise shaping can sustain many more different LCs than SDMs (of equivalent order) with mild noise shaping, and are also more robust against dithering the quantizer. Also, it has been shown that LCs of a long period, even though the number of LCs grows exponentially, are much more sensitive to a small disturbance than a short LC. Likewise, it can be proven that SDMs with aggressive noise shaping are more sensitive to small disturbances than mildly noise shaping SDMs—even though the latter exhibit a much smaller number of sustainable LCs. This is corroborated by the general experimental observation that aggressive noise shapers are less susceptible to LCs than mild noise shapers. As a result, dithering the quantizer as a means to remove LCs should be discouraged. Adding (even tiny) amounts of, perhaps shaped, noise to the input of the SDM is far more effective in achieving the same goal.

It should be noted that all the results and observations presented here apply only to feedforward SDMs with constant input. However, the technique may be generalized to other structures, and feedback SDMs have also been investigated by the authors [7]. The use of constant input is a highly relevant situation, since these LCs are known to be problematic and easily observed. However, SDMs are intended to be used primarily with input signals of a limited bandwidth. The dynamic behavior of SDMs under periodic or noisy input is still an unknown area, and is currently a research topic pursued by the authors.

Appendix A
Proof That Equalities are a Sufficient Condition for a Limit Cycle

In this appendix, we will prove that the set of inequalities (9) leads to the same solution as the set of equalities (5) would, i.e., whether the LC condition observed at the output of the SDM

$$y^{(n+P)} = y^{(n)} \tag{44}$$

leads to the existence of a periodic orbit in state space as known in stability analysis

$$\mathbf{s}^{(n+P)} = \mathbf{s}^{(n)}. \tag{45}$$

For example, if the output sequence happens to be a periodic sequence of period P, one could ask the question whether there exists a possibility that an initial state $\mathbf{s}_0$ does not return to this value after propagation over P cycles, but to a different state vector $\mathbf{s}'$. If this state vector generates the same output sequence again, *etc.*, we have a LC without fulfilment of (5). To that end, we look at the propagation of the state variables (3), after a large number NP of cycles. We will further assume that $(\mathbf{I} - \mathbf{A}^P)$ is invertible; when it is not we will define a new transition matrix $\tilde{\mathbf{A}}$ as

$$\tilde{\mathbf{A}} = \mathbf{A}' + \epsilon\mathbf{I} \tag{46}$$

where $\mathbf{A}'$ is the original transition matrix, and $\mathbf{I}$ is the unit matrix. Now $(\mathbf{I} - \tilde{\mathbf{A}}^P)$ is invertible by definition; at the end of the analysis we will than have to take the limit $\epsilon \to 0$ to obtain the final result.

From (3), we subsequently determine the states $\mathbf{s}^{(NP)}$ at the NPth clock cycle as

$$\mathbf{s}^{(NP)} = \tilde{\mathbf{A}}^{NP}\mathbf{s}^{(0)} + \left[\sum_{i=0}^{NP-1}(u^{(i)} - y^{(i)})\tilde{\mathbf{A}}^{NP-i-1}\right]\mathbf{d}. \tag{47}$$

The summation can be written as two nested summations

$$\mathbf{s}^{(NP)} = \tilde{\mathbf{A}}^{NP}\mathbf{s}^{(0)} + \sum_{j=0}^{N-1}\left[\sum_{i=0}^{P-1}(u^{(i+jP)} - y^{(i+jP)})\tilde{\mathbf{A}}^{NP-(jP+i)-1}\right]\mathbf{d}. \tag{48}$$

Because $\{y^{(i)}\}$ is a LC and $\{u^{(i)}\}$ is constant, we have, by definition, $u^{(i+P)} - y^{(i+P)} = u^{(i)} - y^{(i)}$, and thus we can write

$$\mathbf{s}^{(NP)} = \tilde{\mathbf{A}}^{NP}\mathbf{s}^{(0)} + \sum_{j=0}^{N-1}\left[\sum_{i=0}^{P-1}(u^{(i)} - y^{(i)})\tilde{\mathbf{A}}^{NP-(jP+i)-1}\right]\mathbf{d}. \tag{49}$$

The terms not dependent on i can be moved outside the summation

$$\mathbf{s}^{(NP)} = \tilde{\mathbf{A}}^{NP}\mathbf{s}^{(0)} + \left[\sum_{j=0}^{N-1} \tilde{\mathbf{A}}^{(N-1)P-jP} \sum_{i=0}^{P-1}(u^{(i)} - y^{(i)})\tilde{\mathbf{A}}^{P-i-1}\right]\mathbf{d} \quad (50)$$

and the summation over j can be written more simply as

$$\mathbf{s}^{(NP)} = \tilde{\mathbf{A}}^{NP}\mathbf{s}^{(0)} + \left[\sum_{j=0}^{N-1} \tilde{\mathbf{A}}^{jP} \sum_{i=0}^{P-1}(u^{(i)} - y^{(i)})\tilde{\mathbf{A}}^{P-i-1}\right]\mathbf{d}. \quad (51)$$

Because the sum over i represents a constant, we define

$$\mathbf{L}_P(\{y^{(i)}\}) = \left[\sum_{i=0}^{P-1}(u^{(i)} - y^{(i)})\tilde{\mathbf{A}}^{P-i-1}\right]\mathbf{d} \quad (52)$$

in line with earlier definitions in Section II.B. With this definition, we can write (49) concisely as

$$\mathbf{s}^{(NP)} = \tilde{\mathbf{A}}^{NP}\mathbf{s}^{(0)} + \sum_{j=0}^{N-1} \tilde{\mathbf{A}}^{jP}\mathbf{L}_P(\{y^{(i)}\}). \quad (53)$$

Realizing that the finite sum represents a geometric series, and because we have defined $\tilde{\mathbf{A}}$ such, that $(\mathbf{I} - \tilde{\mathbf{A}}^P)$ is by definition invertible, (53) can be expressed as follows:

$$\mathbf{s}^{(NP)} = \tilde{\mathbf{A}}^{NP}\mathbf{s}^{(0)} + (\mathbf{I} - \tilde{\mathbf{A}}^{NP})(\mathbf{I} - \tilde{\mathbf{A}}^{P})^{-1}\mathbf{L}_P(\{y^{(i)}\}). \quad (54)$$

For convenience, we will further define

$$\mathbf{r} = (\mathbf{I} - \tilde{\mathbf{A}}^{P})^{-1}\mathbf{L}_P(\{y^{(i)}\}) \quad (55)$$

due to which we can further compress (53) as

$$\mathbf{s}^{(NP)} = \tilde{\mathbf{A}}^{NP}(\mathbf{s}^{(0)} - \mathbf{r}) + \mathbf{r}. \quad (56)$$

In order for a LC to be stable, $\lim_{N\to\infty}|\mathbf{s}^{(NP)}|$ must be bounded. We will discriminate two cases.

A. Case 1: $(\mathbf{s}^{(0)} - \mathbf{r})$ *Corresponds to Eigenvalues With Norm Larger Than 1*

If $(\mathbf{s}^{(0)} - \mathbf{r})$ contains directions which correspond to eigenvalues with a norm larger than 1 of $\tilde{\mathbf{A}}$, $\lim_{N\to\infty}|\mathbf{s}^{(NP)}|$ is bounded *only* when

$$\mathbf{s}^{(0)} = \mathbf{r} \quad (57)$$

leading to

$$\mathbf{s}^{(NP)} = \mathbf{s}^{(0)} \quad (58)$$

which is the conjecture on which the results in Section II.B are based.

Upon applying the definition of (46) to (56), we see that we can safely take $\lim_{\epsilon\to 0}$ by setting $\epsilon = 0$, as a result of which (57) is independent of the invertibility of $(\mathbf{I} - \mathbf{A})$. The definition of $\mathbf{r}$ in (55) also turns out to be a familiar result, as with $\mathbf{r} = \mathbf{s}^{(0)}$ it is identical to (6) in Section II.B.

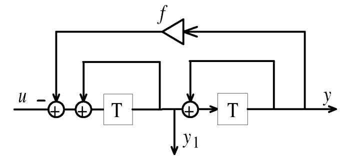

Fig. 11. Single resonator section, as used in an SDM.

B. Case 2: $(\mathbf{s}^{(0)} - \mathbf{r})$ *Corresponds to Eigenvalues With Norm Smaller Than 1*

If, on the other hand, $(\mathbf{s}^{(0)} - \mathbf{r})$ contains only directions which correspond to eigenvalues < 1 of $\tilde{\mathbf{A}}$, these directions will be reduced to zero if $\lim_{N\to\infty}$. We thus obtain as a result

$$\mathbf{s}^{(NP)} = \mathbf{r} \quad (59)$$

which is identical to (57) when $\mathbf{s}^{(NP)} = \mathbf{s}^{(0)}$. However, we do not have the result that $\mathbf{s}^{(NP)} = \mathbf{s}^{(0)}$. Thus, in this case, we have a possibility that an initial state $\mathbf{s}^{(0)}$ does not return to this value after propagation over P cycles, but to a different state vector $\mathbf{s}'$, and that this state vector generates the same output sequence again. After a long number of LC periods, however, $\mathbf{s}^{(NP)}$ converges to a unique value, such that $\mathbf{s}^{(NP)} - \mathbf{s}^{((N+1)P)} = 0$ which is a situation identical to Case 1.

APPENDIX B
SOLUTION TO THE DIFFERENCE EQUATIONS OF A RESONATOR SECTION

In order to be able to determine the effects of a resonator section in an SDM, we will solve the difference equation describing such a system. Whereas a linear algebraic approach, such as used in the main text of the paper, will give the same results, it proves to be a rather rather cumbersome excercise, for which reason we refrain from presenting this approach. The resonator system that we study, is depicted in Fig. 11. The input to the system is u, and the system output that we will study, is labeled y in Fig. 11. The output labeled y_1 can be obtained in an almost identical way as described below.

The difference equations describing the resonator output $y^{(n)}$ are given by

$$y^{(n)} - 2y^{(n-1)} + (1 + f)y_{(n-2)} = u^{(n-2)} \quad (60)$$

different realizations of a resonator section (for example, with nondelayed integrators *etc.*) will have a slightly different but comparable difference equation. We will seek the solution to (60) as

$$y^{(n)} = Ae^{\beta n} + \phi^{(n)} \quad (61)$$

where $\phi^{(n)}$ is the particular solution.

We will now first want to find the homogeneous solution $h^{(n)} = Ae^{\beta n}$, that is, the solution to the substitution of $h^{(n)}$ in (60)

$$Ae^{\beta n}[1 - 2e^{-\beta} + e^{-2\beta}(1+f)] = 0 \quad (62)$$

which is identical to

$$(1-e^{-\beta})^2 + fe^{-2\beta} = 0. \quad (63)$$

Solving for $h^{(n)}$, we obtain two independent solutions $h_{1,n}$ and $h_{2,n}$ to $h^{(n)}$

$$h_1^{(n)} = A\frac{1}{1+i\sqrt{f}} = A(\sqrt{1+f})^n e^{+i\arctan(\sqrt{f})}$$
$$h_2^{(n)} = B\frac{1}{1-i\sqrt{f}} = B(\sqrt{1+f})^n e^{-i\arctan(\sqrt{f})}. \quad (64)$$

This homogeneous solution can also be obtained rather easily from the algebraic approach, as the eigenvalues from the transition matrix describing the resonator are given by $\lambda = 1 \pm j\sqrt{f} = \sqrt{1+f}e^{\pm i\arctan(\sqrt{f})}$.

To obtain the particular solution $\phi^{(n)}$, we will substitute $\phi^{(n)}$ in (60). This results in

$$\phi^{(n)} - 2\phi^{(n-1)} + \phi^{(n-2)} + f\phi^{(n-2)} = u^{(n-2)}. \quad (65)$$

For slowly varying $\phi^{(n)}$, this can be approximated by

$$\phi''^{(n)} + f\phi^{(n)} \approx u^{(n)} \quad (66)$$

where $\phi''^{(n-1)}$ is the second derivative of ϕ. A solution for $\phi^{(n)}$ can now be found when both $u^{(n)}$ and $\phi^{(n)}$ are expanded in a Fourier series

$$u^{(n)} = \sum_k a_k \cos(\omega_k n) + \sum_l b_l \sin(\omega_l n)$$
$$\phi^{(n)} = \sum_k a'_k \cos(\omega_k n) + \sum_l b'_l \sin(\omega_l n). \quad (67)$$

When these Fourier expansions are substituted in (66), we obtain the Fourier coefficients a'_k and b'_k

$$a'_k = \frac{a_k}{f-\omega_k^2}; \quad b'_k = \frac{b_k}{f-\omega_k^2}. \quad (68)$$

For example, if $u^{(n)}$ is a constant, or a slowly varying function, a good approximation for $\phi^{(n)}$ is given by

$$\phi^{(n)} = \frac{u^{(n)}}{f}. \quad (69)$$

The general solutions $y^{(n)}$ to (60) now read

$$y^{(n)} = A(\sqrt{1+f})^n e^{i\arctan(\sqrt{f})} + B(\sqrt{1+f})^n e^{-i\arctan(\sqrt{f})} + \phi^{(n)}. \quad (70)$$

It is the boundary conditions that will determine the constants A and B.

The first obvious requirement is that $y^{(n)}$ be real, that is, $A = \pm B$. Further we have that either

$$y_0 = 0; \quad u_0 \neq 0 \quad (71)$$

when a step function is applied to the input, *or*

$$u^{(0)} = y^{(0)} = 0;$$
$$y^{(1)} = 0; \quad \forall u^{(1)} \quad (72)$$

when a gradually increasing function is applied to the input. It is easily verified that boundary condition (71) results in

$$y^{(n)} = -\phi^{(0)}\cos(n\arctan(\sqrt{f}))(\sqrt{1+f})^n + \phi^{(n)}; \quad n \geq 0 \quad (73)$$

while the second set of boundary conditions (72) results in

$$y^{(n)} = -\frac{\phi^{(1)}}{\sin(\arctan(\sqrt{f}))}\sin(n\arctan(\sqrt{f})) \cdot (\sqrt{1+f})^{n-1} + \phi^{(n)}; \quad n \geq 1. \quad (74)$$

The first solution (73) is more realistic in the sense that it is very unlikely that $u_0 = 0$, and is the solution that leads to (41) in the main text upon the assumption that $u^{(n)}$ varies only slowly, leading to the substitution $\phi^{(n)} \approx u^{(n)}/f$.

Appendix C
Description of Sigma Delta Modulators

The feedforward SDMs used in Section V are all fifth order, and are referred to by a code which gives their corner frequency and a letter "a" if they include resonators, and "b" if not. They are all of the type displayed in Fig. 2

NTF type	code
$f_c = 160$ kHz, with resonators;	160a
$f_c = 120$ kHz, with resonators;	120a
$f_c = 120$ kHz, without resonators;	120b
$f_c = 100$ kHz, with resonators;	100a
$f_c = 80$ kHz, with resonators;	80a
$f_c = 80$ kHz, without resonators;	80b

The design of the SDMs is such that the NTFs of the SDMs are all of the Butterworth high-pass type if $f_i = 0$. For SDM120a, the NTF has a −3 dB point at 120 kHz; for SDM80a, this point is at 80 kHz. Thus, SDM80a represents a much less aggressive noise shaper than SDM120a. The coefficients c_i for these two SDMs are tabulated in the table shown at the bottom of the page.

SDM	c_1	c_2	c_3	c_4	c_5	f_1	f_2
120a	0.863 756	0.361 381	0.090 003	0.013 209	0.000 908	0.0018	0.000 685
80a	0.576 107	0.162 475	0.027 609	0.002 805	0.000 136	0.0018	0.000 685

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