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Dither and noise modulation in sigma delta modulators

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ABSTRACT

In recent years there has been considerable debate over the suitability of 1-bit Sigma-Delta modulation (SDM) for high-quality applications. Much of the debate has centered on whether it is possible to properly dither such a system. It has been shown that dither with a triangular probability distribution should be applied to the quantizer input in a pulse code modulation system. This is not the case for all A/D converters. We show that the dependence of error moments on input is inherently different in sigma delta modulators, and that the effect of dither depends on whether the quantiser is one bit or multibit. These statements are proven for simple SDMs and verified by simulation.

1. INTRODUCTION

In recent years there has been considerable debate concerning some technical issues involving the practical use of sigma delta modulation in the mastering and archiving of audio recordings. The argument has been brought forth that 1-bit Sigma-Delta modulation (SDM) is unsuitable for high-quality applications[1,2]. Part of the argument is based on the premise that dither with a triangular probability distribution function (PDF) eliminates distortion, noise modulation, and other signal dependent artefacts. This is based on theoretical results and psycho-acoustic testing of pulse code modulation (PCM) systems [3]. The theoretical results showed that rectangular dither renders the first conditional moment of the error zero, but triangular (or higher order) dither is necessary to render the second moment independent of the input. This is a requirement for the residual noise to be input independent.

The argument continues with the assertion that in a 1-bit system, the use of 2 LSB triangular dither guarantees that the quantizer is fully loaded even before any signal is applied. However, other authors have pointed out that operating a sigma delta modulator in the overload region of the quantiser is not an uncommon situation[4]. Furthermore, the alternative situation of using high multibit quantizers, is strongly limited by component tolerances [5].

Several authors have suggested that sigma delta modulators may be self-dithered[6,7]. This would have the effect of minimising the need for the introduction of large scale dither into the system. A justification for self dithering has been given[8], and recent results indicate that high order sigma delta modulators behave very much like a well-dithered system[9]. However, the ability of self-dithering to remove artefacts is unclear[10]. Alternatives to dithering have been proposed, including bit-flipping[11,12] and chaotic[13-15] systems. Several authors have discussed possible limitations of chaotic systems [16,17], but these results have yet to be put into the proper framework.

What is clear from this description of recent treatments of dither in sigma delta modulators is that there has been a wealth of analysis of such systems, with seemingly contradictory conclusions. The effects of non-subtractive dither in pulse code modulation systems has been well-studied ([18] and references therein). However, sigma delta modulators employ a feedback loop that affects the probability distribution of the input to the quantizer. These effects can not be foreseen using the arguments in [3]. The purpose of the work presented here is to determine more accurately the dependence of the moments of the error on the input due to the presence of a feedback loop and due to the low-bit rate quantization that is applied in most sigma delta modulation systems.

2. PULSE CODE MODULATION

We begin by looking at dither distributions and the resultant quantization error in PCM systems. The approach used is slightly simpler though less rigorous than that in [3]. The analysis in this section is presented primarily because the analysis of SDM systems can be made via an extension of this approach.

An infinite midtread quantizer has the transfer function $Q(w)=\Delta\lfloor w/\Delta+1/2\rfloor$, where w is the input to the quantizer, and Δ is the quantization step size (least significant bit, or LSB). If x is the input to a PCM system, then the total error is simply

$$e=\Delta\lfloor x/\Delta+1/2\rfloor-x \qquad (1)$$

Thus the m^{th} moment of the error is, for a given x, is

$$\sum e^m p(e)=(\Delta\lfloor x/\Delta+1/2\rfloor-x)^m \qquad (2)$$

Under such circumstances, all error moments are dependent on the input.

However, if rectangular PDF dither of size 1 LSB is applied immediately before quantization then the PDF of the input to the quantizer has the form,

$$p(w)=\begin{cases}1/\Delta & x-\Delta/2< w\le x+\Delta/2\\ 0 & otherwise\end{cases} \qquad (3)$$

The input ranges over 1 LSB, so the output can assume only 2 possible values. If we define, $y=x/\Delta-\lfloor x/\Delta\rfloor$, then the error has the distribution,

$$e=\begin{cases}-\Delta y & p=1-y\\ \Delta(1-y) & p=y\end{cases} \qquad (4)$$

and hence, using (2), only the first error moment is independent of the input.

The use of triangular dither gives the quantizer input PDF the form,

$$p(w)=\begin{cases}(w-x+1)/\Delta & x-\Delta\le w< x\\ (1-w+x)/\Delta & x\le w< x+\Delta\end{cases} \qquad (5)$$

This can be generated by summing two rectangular PDF dithers of width 1 LSB. The input ranges over 2 LSB, so the output can assume only 3 possible values. If we define $y=\lfloor x/\Delta+1/2\rfloor-x/\Delta$ then the error is distributed as

$$e=\begin{cases}\Delta(y-1) & p=(1/2+y)^2/2\\ \Delta y & p=3/4-y^2\\ \Delta(y+1) & p=(1/2-y)^2/2\end{cases} \qquad (6)$$

which makes the first two error moments independent of the input.

3. MULTI-BIT SIGMA DELTA MODULATION

The results for PCM systems do not apply for sigma delta modulators because the input to the quantizer includes the system input and noise shaped error. An undithered first order SDM system feeds back the delayed quantization error and adds it to the input. This has the effect that the average quantized values equals the input. So, for a midtread quantizer with constant input of $x=(n+m)\Delta$, where n is an integer and m is between 0 and 1, $(1-m)^{\text{th}}$ of the time x will be quantized to $n\Delta$, and m^{th} of the time x will be quantized to $(n+1)\Delta$. This gives exactly the same error distribution as for PCM with rectangular dither, Equation (4).

The application of dither to an SDM with infinite quantizer produces some interesting results. We first consider the application of dither at the system input, as shown in Figure 2. Although this is rarely performed in practical applications, it is useful for illustration and serves to demonstrate why dither placed elsewhere in the feedback loop is often more beneficial.

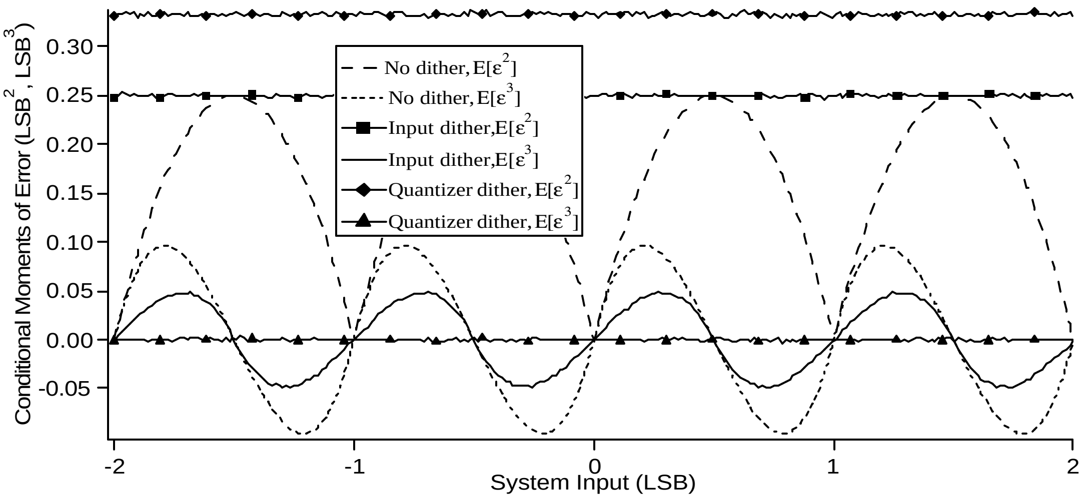

Figure 1. Second and third order moments of error as a function of input for a multibit first order SDM without dither, with rectangular dither at input, or rectangular dither before quantization.

This system is given by the following equation,

$$U(n+1) = x(n) + v(n) + U(n) - Q(U(n)) \qquad (7)$$

The quantization error is thus given by

$$e = \Delta \lfloor U(n)/\Delta + 1/2 \rfloor - U(n) \qquad (8)$$

So quantization error, quantizer output minus quantizer input, is determined by the distribution of $U(n)$. We can use the following: *If the input to the quantizer has a PDF determined by the sum of n uniform PDFs of 1 LSB amplitude (n^{th} order dither), then the quantization error has a uniform PDF of 1 LSB amplitude centred around 0.*

This can be verified by simulation, and can be easily shown for rectangular (n=1) and triangular (n=2) distributions. Thus the input to the quantizer has a distribution determined by a constant x, and the sum of the dither distribution and a rectangular distribution. So the application of rectangular dither to the input of a first order sigma delta modulator gives a total error probability distribution exactly like that of triangular dither applied to a PCM system. This implies that in such a system, only rectangular dither is necessary to make the first and second order conditional moments of error constant with respect to input.

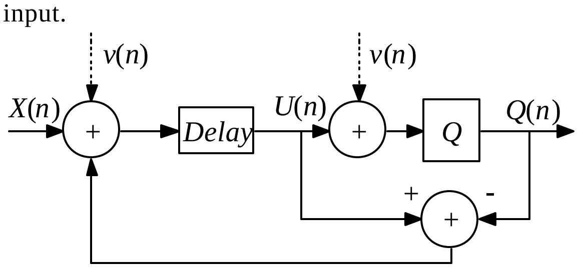

Figure 2. A 1st order SDM with dither applied either to system input or before quantization.

Dither applied at the input to the quantizer, as depicted in Figure 2, has a different behaviour. A dithered first order sigma delta modulator with infinite quantizer is given by Equation (9).

$$U(n+1) = x(n) + U(n) - Q(U(n) + v(n)) \qquad (9)$$

Rewriting (9) in terms of quantization error, we have

$$U(n) = x - e_Q(n-1) - v(n-1)$$

$$e_Q(n-1) = Q(U(n-1) + v(n-1)) - (U(n-1) + v(n-1)) \qquad (10)$$

So, the total error is

$$Q(x - e_Q(n-1) - v(n-1) + v(n)) - x \qquad (11)$$

Thus, if n^{th} order dither is applied to the quantizer input in an SDM system, then the error has the equivalent distribution to a PCM system with $(2n+1)^{th}$ order dither.

These results have been confirmed in simulations with the addition of first, second and third order dither. They indicate the benefits of using an SDM system with dither applied immediately prior to quantization.

Figure 1, the second and third order conditional moments of error are plotted as a function of input for multibit sigma delta modulators without dither, with dither applied to the input, and with dither applied to the quantizer. Each moment was estimated using 100,000 iterations at 250 equally spaced input values between -2 and 2. The moments found are equivalent (with minor differences due to finite iterations and imperfect dithering) to the moments from a PCM system with rectangular dither, triangular dither, and 3rd order dither respectively.

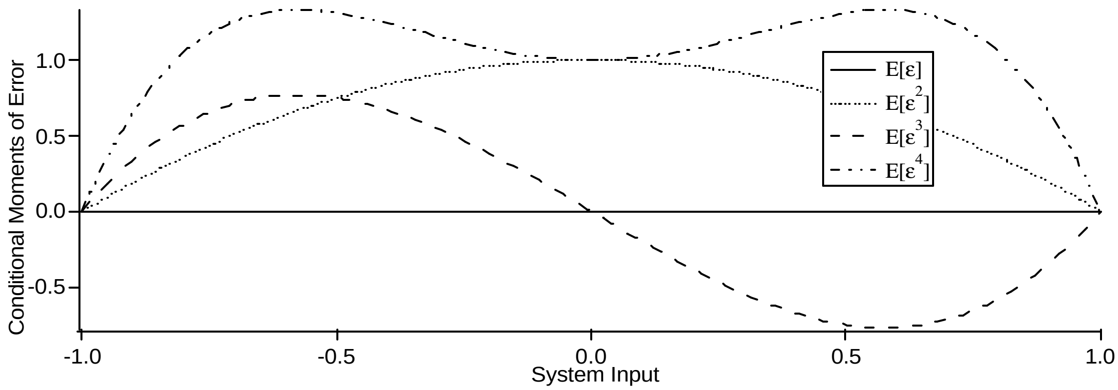

Figure 3. The first four conditional moments of error for an SDM with a single bit +/- 1 quantizer.

As predicted by [3] for the equivalent PCM error distributions, SDM without dither shows dependence on the input for second and third order moments. The input dependence in the second order moment is eliminated if rectangular dither of width 1 LSB is applied to the system input, and input dependence in both moments is eliminated if rectangular dither is applied to the quantizer input. In all three situations, the first order (average) conditional moment of error is a constant zero.

4. 1-BIT SIGMA DELTA MODULATION

A requirement of most SDM systems is that average output must equal average input. That is, for constant input *x*,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}Q(n)=x \tag{12}$$

where $Q(n)$ is the quantized output of the modulator. This is regardless of the order and structure of the modulator.

For multibit systems, this allows a variety of conditional probability distributions for the quantized output. But for a single bit quantizer, this forces the probabilities to obey

$$p(Q(n)=1)\cdot 1+p(Q(n)=-1)\cdot(-1)=x \tag{13}$$

So that in the limit of an infinite number of samples from such a probability distribution, (12) holds. (13) can be rephrased in terms of the total error as

$$e=\begin{cases}1-x & p=(x+1)/2\\ -1-x & p=(1-x)/2\end{cases} \tag{14}$$

Thus the first moment is 0, and the m^{th} moment is an m^{th} order polynomial. All 1-bit sigma delta modulators with a +/-1 quantizer must have this error distribution. This is confirmed by Figure 3, which depicts the first four conditional moments of error for a 1 bit SDM. These results actually represent six simulated systems: 1st and 2nd order sigma delta modulators, each without dither, with rectangular dither at the quantizer, and with triangular dither at the quantizer. The inclusion of dither only served to make the distribution more noisy, but did not change the structure.

5. CONCLUSION

The results clearly show that the application of dither in multibit sigma delta modulators successfully eliminates much of the input dependence in the error. In fact, SDM systems outperform PCM systems since a much simpler form of dither may be used. However, the addition of dither has *no effect* on the conditional moments of error in most 1-bit sigma delta modulators. This is regardless of the probability distribution of the dither. Thus, the high performance of single bit SDMs in terms of noise modulation, distortion and other signal dependent artefacts must be explained by other means.

The results were verified using numerous simulations, and analyses of the error distributions as well as the error moments. They were also justified using heuristic arguments. However, they have not yet been put on a firm mathematical foundation. Furthermore, the full effects of high order noise shaping in regards to the error PDFs was not investigated. These two areas are the focus of the authors' ongoing research.

6. ADDENDUM

Since the preparation of this work, the authors have discovered that many of the issues presented here have been addressed in {Lipshitz, 1993 #12127;Wannamaker, 1993 #12128}. The methods used differed from the methods herein, thus they reached some different conclusions (although they also showed that the presence of feedback greatly changes the error distribution). However, that work was put on a firm mathematical basis, and was a logical extension of their work on PCM systems, whereas the work presented here is more compact, and is focused on the issues brought up in the recent debate on dither and SDMs.

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