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A Modal Treatment of McCain-Turner

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The Basic Theory

McCain and Turner

- language $\mathfrak{L}$
- a collection D of "causal rules"

$$\phi_i \Rightarrow \psi_i$$

- Given any model I of $\mathfrak{L}$, define

$$D^I \quad = \quad \{\psi | \phi \Rightarrow \psi \in D, \ I \models \phi\} \tag{1}$$

- say that a model I is *causally explained* if I is the only model of D^I
- say that P is a *consequence* of D iff P is true in every causally explained model of $\mathfrak{L}$.

Models

- Can regard a causal theory as a map from *models of* $\mathfrak{L}$ to *sets of models of* $\mathfrak{L}$
- that is, a *relation* on the set of models of *lang*
- almost a Kripke model, except that $\mathfrak{L}$ is not modal
- but we can fix that
- add a modal operator $\Box$ to $\mathfrak{L}$
- semantics:
  - set of worlds = models of $\mathfrak{L}$
  - accessibility relation given by $I \mapsto$ models of I^D
- then, in this model, $I \Vdash \Box P$ iff $I^D \vdash P$
- if $\phi \Rightarrow \psi \in D$, then $\phi \rightarrow \Box\psi$ is true at every world of this model.

Proof Theory

How do we axiomatise this?

- We clearly have theorems $\vdash \phi \to \Box\psi$ ($\phi \Rightarrow \psi \in D$)
- More generally,

$$\frac{\Gamma \vdash \phi_1 \wedge \ldots \wedge \phi_k, \Delta \quad \psi_1, \ldots, \psi_k \vdash X}{\Gamma \vdash \Box X, \Delta}$$

- But this doesn't constrain $\Box$ enough: these rules are sound for any model with $\Box P$ forced by all worlds, for any proposition P.
- we need *only those* $\Box P$ to be true which are compelled to be so by the above rule
- we can do this by adding the corresponding left rule:

$$\frac{\{\Gamma, \phi_1, \ldots, \phi_k \vdash \Delta \quad \psi_1, \ldots, \psi_k \vdash X\}_{\text{all such}}}{\Gamma, \Box X \vdash \Delta}$$

Good News

- We can prove cut elimination
- (use results of Schroeder-Heister)

Bad News

- the left rule can be infinitary
- or, even when finite, undecidable
- but only so when the original procedure is similarly bad
- it is well behaved for “standard” applications
- even when the left rule is infinitary, the system is metatheoretically very powerful
- (very like inifinitary systems of arithmetic)

What We Can Prove

- the Kripke model we constructed above is the canonical model of this system
- X is causally explained iff there are propositions $P_i, \ldots, P_k$, $Q_1, \ldots, Q_l$ such that

$$P_1 \to \Box P_1, \ldots, P_k \to \Box P_k, \Box Q_1 \to Q_1, \ldots, \Box Q_l \to Q_l \vdash X$$

Applications

A *standard causal theory* is something of the following form

1. $f_0 \Rightarrow \Box f_0$ and $\neg f_0 \Rightarrow \Box \neg f_0$, for any fluent f at time 0;
2. $a_t \Rightarrow \Box a_t$ and $\neg a_t \Rightarrow \Box \neg a_t$, for any action a at any time t;
3. $f_{t-1} \wedge f_t \Rightarrow \Box f_t$ and $\neg f_{t-1} \wedge \neg f_t \Rightarrow \Box \neg f_t$, for any fluent f and any time t;
4. $f_{t-1} \wedge a_{t-1} \Rightarrow \Box g_t$, for any time t, where f is the precondition and g is the postcondition of action a.
5. $\neg P \Rightarrow \bot$, for any domain constraint P.

Example 1: using the constraints

$$
\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, C \vdash \Box A, \Delta}}{\Gamma \vdash \neg C, \Box A, \Delta}\,\neg\mathrm{R} \qquad \dfrac{}{\bot \vdash A}\,\bot}{\dfrac{\Gamma \vdash \Box A, \Box A, \Delta}{\Gamma \vdash \Box A, \Delta}\,\mathrm{RC}}\,\Box\mathrm{R}
$$

Example 2: TrajectoriesSuppose $X = \bigwedge_{t=0}^{t_1} X_t$ is a *trajectory*:that is

- each X_t is a conjunction of fluents at t and actions at t
- the actions at t, when applied to the fluents at t, yield the actions at $t+1$

Then:

- $X \vdash \Box X$
- proof:
  - for each t we have a set $psi_1, \ldots, \psi_k$ of heads of causal rules with
  $$\psi_i, \ldots, \psi_k \vdash X_t$$
  - and
  $$X_{t-1}, X_t \vdash \phi_1 \wedge \ldots \wedge \phi_k$$
  where the ϕ_i are the corresponding bodies
  - So we have
  $$\{\text{all } \psi s\} \vdash \bigwedge_{t=0}^{t_1} X_t$$
  - and we get the result by a gigantic application of □R.

Example 3: CoalgebrasSuppose we have a proposition A such that

$$C, A \vdash \Box A$$

where A is the constraints.

Call this a $\Box$-coalgebra.

Suppose also that

$$C, A \vdash f_{t_1}$$

where f_{t_1} is a fluent at some positive time t_1.Then $A \vdash C \rightarrow f_{t_1}$, so

$$\Box A \vdash \Box(C \rightarrow f_{t_1})$$

and so

$$C, A \vdash \Box(C \rightarrow f_{t_1})$$

Look at a cut free proof of this.

- we may get a lot of branching, due to the structure of C
- but we have to apply $\Box R$ to the right hand side at least once on each branch
- and when we apply it, we will get either
  - $f_{t_1 - 1} \wedge f_{t_1}$, or
  - $g_{t_1 - 1} \wedge a_{t_1 - 1}$, where a is an action and g is its preconditions, or

- some conjunction of the above, or
- the negated constraints(but if the branch ends there, the constraints + A must entail $\neg f_{t_1}$, which is a trivial case)

So, finally, for some set I which indexes the branches:

$$C, A \vdash \bigvee_I \text{stuff}_i$$

where, for each $i \in I$, stuff$_i$ is some conjunction of fluents at $t_1 - 1$, actions at $t_1 - 1$, and fluents at t_1.

So we can now get an entailment

$$C, A \vdash \bigvee_I (\text{fluents at } t_1 - 1)_i$$

and we can continue the induction until $t = 0$.

So our coalgebra, A, entails something which "looks like" a disjunction of trajectories.

Example 4: invariants

Suppose that we have an *invariant*, $P(t)$: that is, something with the properties that:

- if $f_t \vdash P_t$, then $f_{t-1} \vdash P_{t-1}$
- if $f_t \vdash P_t$, and if there is an action a_{t-1} which acts on g_{t-1} to give f_t, then $g_{t-1} \vdash P_{t-1}$,

then (under suitable assumptions: e.g. no constraints)

$$\mathbb{P} = \bigvee_{t=0}^{t_1} P_t$$

is a $\Box$-algebra: i.e.

$$\Box P \vdash P$$

To Do

- use the metatheory to find axiomatisations with better properties
- in particular, look at time reversed axioms
- connection with automata theory