Motivation 00000

Final Coalgebra for *powerset*

Further research

Relational and final coalgebra semantics for dynamic logics

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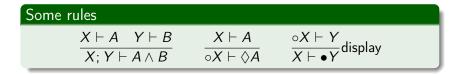
ALCOP 2014, London, UK

Informal presentation of Display Calculi

Display calculi: variation of sequent calculi

Why: they are modular, cut elimination meta theorem

How do they work? 1 property = 1 rule



; structural symbols \longrightarrow to manipulate structures \land operational symbols \longrightarrow formulas such as $\Diamond A$ are "frozen".

Main feature: display property

Informal presentation of Display Calculi

Display calculi: variation of sequent calculi

Why: they are modular, cut elimination meta theorem

How do they work? 1 property = 1 rule

Some rules		
$X \vdash A Y \vdash B$	$X \vdash A$	$\circ X \vdash Y$
$X; Y \vdash A \land B$	$\overline{\circ X \vdash \Diamond A}$	$\frac{\circ X \vdash Y}{X \vdash \bullet Y} display$

; structural symbols \longrightarrow to manipulate structures

 \land operational symbols \longrightarrow formulas such as $\Diamond A$ are "frozen".

Main feature: display property

display property \Longrightarrow cut elimination meta theorem

Example: The case of EAK

EAK: Epistemic Action and Knowledge

- Syntax: $\varphi := p \mid \neg \varphi \mid \varphi \lor \varphi \mid \diamondsuit \varphi \mid \langle \alpha \rangle \varphi$
- Intuitive meaning: $\Diamond_a \varphi$ agent *a* thinks that φ is possible and

 $\langle \alpha \rangle \varphi$: α can be executed, and after that, φ holds.

Protocol to design the proof system of a given logic:

1/ Algebraic description of the logic,

2/ Existing blocks: intuitionistic/classical logic, normal modal logic

3/ Design of the derivation rules following some conditions.

Aim: Display rules for each connective **Soundness?** We need adjunction

Remark: $\langle \alpha \rangle$ distributes over joins \rightsquigarrow Does it have an adjoint?

EAK: Frame semantic

Kripke model: $a \in Agent$, $\alpha \in Action$, $p \in Prop$

 $\mathbb{M} := (W, R_{a} \subseteq W \times W, V) \text{ or } \mathbb{M} := (W, \sigma_{a} : W \longrightarrow \mathcal{P}W, V)$

Epistemic update: public announcement $\alpha \cong p$ $\langle \alpha \rangle \varphi$: after *p* is announced, φ holds.

 $\mathbb{M} = (W, R \subseteq W \times W, V) \rightsquigarrow_{\alpha} \mathbb{M}_{\alpha} = (W_{\alpha}, R_{\alpha} \subseteq W_{\alpha} \times W_{\alpha}, V_{\alpha})$

$$W_{\alpha} := V(p) \cap W, \quad R_{\alpha} := R \cap W_{\alpha} \times W_{\alpha}, \quad V_{\alpha}(p) := V(p) \cap W_{\alpha}.$$

/!\ We lose information about \mathbb{M} , some worlds disappear.

Problem: We cannot talk about the inverse of α ! We don't know how to talk about the adjoint? We need them to define our proof system!

Why is the Final Coalgebra interesting?

What is the final coalgebra \mathbb{Z} ?

Idea: "We put all the models together and build a gigantic model."

Advantages:

- dynamic modalities are maps on \mathbb{Z} .
- We do not need to compute a new model anymore.
- We can interpret the adjoint.

Final Coalgebra: formal definition

Let \mathbb{C} be a category, F be an endofunctor on \mathbb{C} . A *F*-coalgebra is a pair $(A, \sigma : A \longrightarrow FA)$ with $A \in \mathbb{C}$. A coalgebra morphism from (A, α) to (B, β) is a arrow $h : A \longrightarrow B$ in \mathbb{C} such that $\beta \circ h = F(h) \circ \alpha$.

 $F-Coalg(\mathbb{C})$: the category of *F*-coalgebras and their morphisms.

Final object

A *final object* in the category \mathbb{C} is an object Z s.t. for any $Y \in C$, there is a unique arrow $f : Y \longrightarrow Z$.

Final Coalgebra

A *final coalgebra* \mathbb{Z} for the functor F is a final object of the category $F-Coalg(\mathbb{C})$.

When does the final coalgebra exist for a Set-endofunctor?

The Final Coalgebra Theorem, Aczel & Mendler

Every set-based functor has a final coalgebra.

Category of classes: objects are classes and arrows are functions between classes.

An endofunctor F is called **set-based** if for each class A and each $a \in FA$, there is a set $A_0 \subseteq A$ and $a_0 \in FA_0$ such that $a = F(i_{A_0,A})(a_0)$, where $i_{A_0,A}$ is the inclusion map $A_0 \hookrightarrow A$.

Idea: F must be entirely defined by what it does on sets.

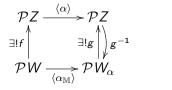
Final coalgebra semantics for powerset functor

A. Baltag, A Coalgebraic Semantics for Epistemic Programs, 2003.

Kripke model: $a \in Agent$, $\alpha \in Action$, $p \in Prop$ $\mathbb{M} := (W, \sigma_a : W \longrightarrow \mathcal{P}W, V)$

Updated model: $\mathbb{M}_{\alpha} := (W_{\alpha}, \sigma'_{a} : W_{\alpha} \longrightarrow \mathcal{P}W_{\alpha}, V_{\alpha})$

Final Coalgebra: $\mathbb{Z} := (Z, \zeta_a : Z \longrightarrow \mathcal{P}Z, V)$



$$[\![\phi]\!]_{\mathbb{M}} = f^{-1}[\![\phi]\!]_{\mathbb{Z}}$$

What about other dynamic logics?

Other logics:

- Monotone modal logic: $\mathbb{M} := (W, \sigma : W \longrightarrow \mathcal{PPW})$
- Intuitionistic modal logic: coalgebras over poset

- ...

Questions about monotone modal logic:

- Which action structure should we use?
- How to define the updated model?
- When does the final coalgebra exist?

Conjecture: Every poset-based functor has a final coalgebra.

Final Coalgebra Theorem for *Poset*?

Conjecture: Every poset-based functor has a final coalgebra.

Category of partially ordered classes: objects are classes with a partial order and arrows are order-preserving functions.

An endofunctor F is called **poset-based** if for each **partially ordered class** (A, \leq) and each $a \in FA$, there is a **poset** (A_0, \leq_0) and $a_0 \in FA_0$ such that $A_0 \subseteq A$ and $a = F(e_{A_0,A})(a_0)$, where $e_{A_0,A}$ is an order-embedding $(A_0, \leq_0) \hookrightarrow (A, \leq)$.

Idea: The class (A, \leq) must be entirely defined by the sub-posets $\{(A_i, \leq_i)\}$. That is why we ask for an order-embedding *e*.

 $\forall a_0, b_0 \in A_0, \quad a_0 \leq b_0 \quad \text{ iff } \quad e(a_0) \leq e(b_0)$

Conclusion

We know:

- Final coalgebra theorem for Sets
- Final coalgebra semantics for EAK
- A display type proof system for EAK

We want:

- Final coalgebra theorem for Posets
- Epistemic monotone modal logic (\diamond and/or $\langle \alpha \rangle$ monotone). Semantics? Axiomatization?

- A display type proof system for monotone dynamic logics, coalgebraic logics, ...

Giuseppe is going to talk about:

Display calculi for dynamic logics