work in progress...

On the Functional Interpretation of Functional Types

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Base types

$\mathbb{N}$	$\simeq$	natural numbers
$\mathbb{B}$	$\simeq$	Booleans (true/false)
$\mathbb{R}$	$\simeq$	real numbers
X	$\simeq$	metric/Hilbert/normed... space

Functional types $\rho \to \tau$ for types ρ, τ

$\mathbb{N} \to \mathbb{N}$	$\simeq$	sequences of natural numbers
$X \to X$	$\simeq$	Self-maps on X
$(\mathbb{N} \to \mathbb{R}) \to \mathbb{R}$	$\simeq$	maps from sequence of reals to a real

Quantifications over base and functional types

$\forall x^X \ldots$	$\simeq$	for each point x in $X \ldots$
$\exists \alpha^{\mathbb{N} \to \mathbb{R}} \ldots$	$\simeq$	there exists a sequence of reals $\alpha \ldots$

Functional Interpretations

formula A $\xrightarrow{I_{\mathcal{F}}}$ *functional specification* $I_{\mathcal{F}}(A)$

proof π_A *of A* $\xrightarrow{I_{\mathcal{P}}}$ *functional* $I_{\mathcal{P}}(\pi_A) \in I_{\mathcal{F}}(A)$

$$I_{\mathcal{F}}(A) = \langle x^{\rho};\, y^{\tau};\, \underbrace{|A| \subseteq \rho \times \tau}\rangle$$

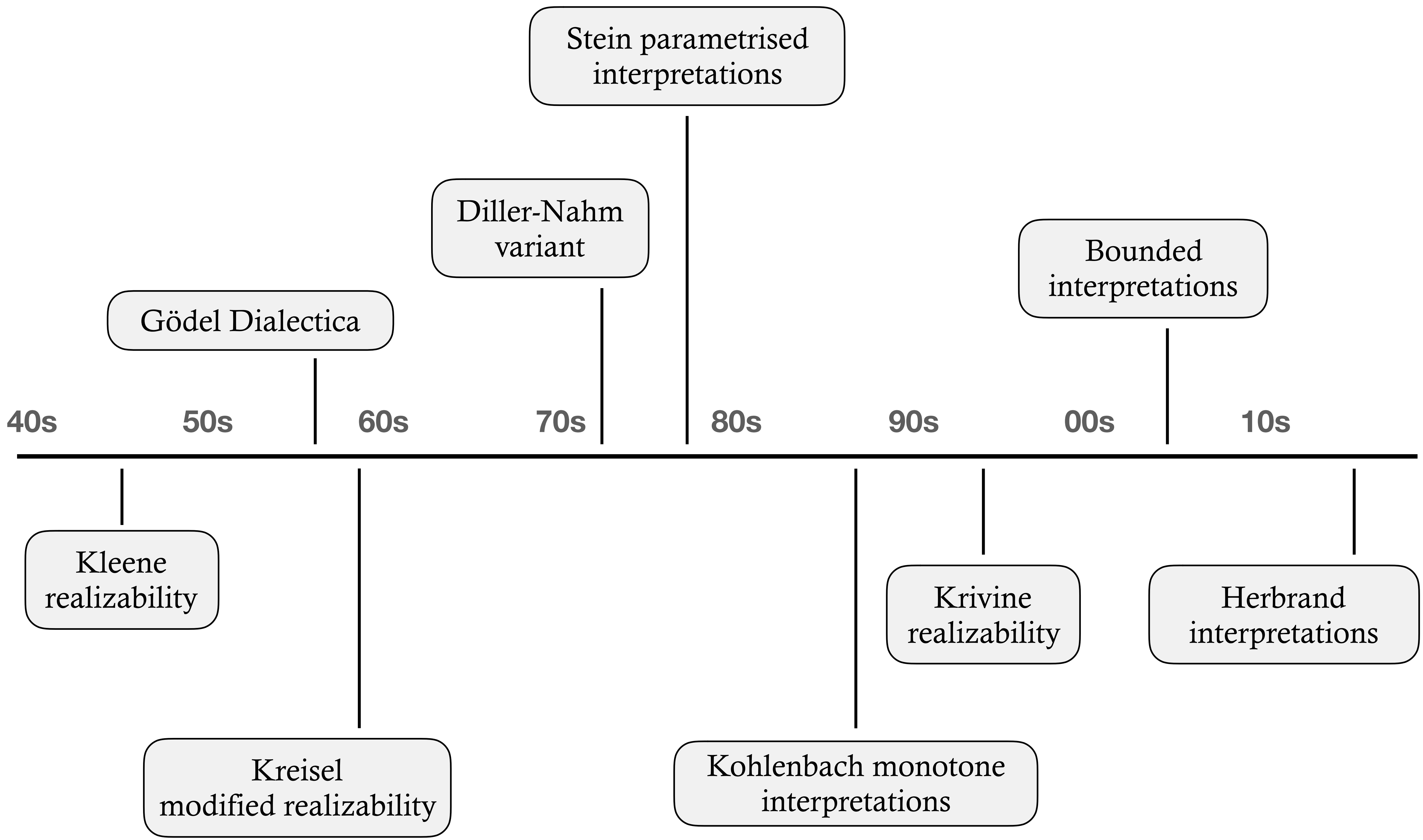
$$|A|^x_y$$

argument

counter-argument

$$I_{\mathcal{F}}(\forall n^{\mathbb{N}} \exists p^{\mathbb{N}} > n \,\mathrm{Prime}(p))$$

$$= \langle f^{\mathbb{N}\to\mathbb{N}};\, n^{\mathbb{N}};\, fn > n \wedge \mathrm{Prime}(fn)\rangle$$


Stein parametrised interpretations
Diller-Nahm variant
Bounded interpretations
Gödel Dialectica
40s
50s
60s
70s
80s
90s
00s
10s
Kleene realizability
Krivine realizability
Herbrand interpretations
Kreisel modified realizability
Kohlenbach monotone interpretations

ÜBER EINE BISHER NOCH NICHT BENÜTZTE ERWEITERUNG DES FINITEN STANDPUNKTES

von Kurt GÖDEL, Princeton

Dialectica, vol. 12, 1958

$$
\begin{aligned}
|A \wedge B|^{x,v}_{y,w} &\equiv |A|^{x}_{y} \wedge |B|^{v}_{w} \\
|A \vee B|^{x,v,b}_{y,w} &\equiv (b=0 \wedge |A|^{x}_{y}) \vee (b \neq 0 \wedge |B|^{v}_{w}) \\
|A \rightarrow B|^{f,g}_{x,w} &\equiv |A|^{x}_{g(x,w)} \rightarrow |B|^{f(x)}_{w} \\
|\forall z^{\tau} A(z)|^{f}_{y,z} &\equiv |A(z)|^{f(z)}_{y} \\
|\exists z^{\tau} A(z)|^{x,z}_{y} &\equiv |A(z)|^{x}_{y}
\end{aligned}
$$

1. If $\mathrm{HA}^{\omega} \vdash A$ then $\mathrm{T} \vdash |A|^{t}_{y}$, for t in system T
2. $\mathrm{HA}^{\omega} + \mathrm{IP}_{\forall} + \mathrm{MP} + \mathrm{AC} \vdash A \leftrightarrow \exists x \forall y |A|^{x}_{y}$

Functional Interpretations of $\rho \to \tau$

(the problem)

$$|A \to B|^{f,g}_{x,w} \equiv |A|^{x}_{g(x,w)} \to |B|^{f(x)}_{w}$$

$$|\forall z^{\tau} A(z)|^{f}_{y,z} \equiv |A(z)|^{f(z)}_{y}$$

$$|\exists z^{\tau} A(z)|^{x,z}_{y} \equiv |A(z)|^{x}_{y}$$

Negative witnesses in a premise become positive witnesses

Universal quantifiers are "negatively" witnessed

Good news:

Markov Principle:

$$\neg \forall x A_{\mathrm{qf}} \to \exists x \neg A_{\mathrm{qf}}$$

Bad news:

Extensionality as a "rule"

$$\frac{\Gamma_{\mathrm{qf}} \vdash fx =_{\rho} gx}{\Gamma_{\mathrm{qf}} \vdash \phi f =_{\tau} \phi g}$$

Bounded functional interpretation

Fernando Ferreira[a,*], Paulo Oliva[b]

Annals of Pure and Applied Logic 135 (2005) 73–112

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^x_y \wedge |B|^v_w$$

$$|A \vee B|^{x,v}_{y,w} \equiv (\forall y' \leq^* y\, |A|^x_{y'}) \vee (\forall w' \leq^* w\, |B|^v_{w'})$$

$$|A \to B|^{f,g}_{x,w} \equiv \forall y \leq^* g(x,w)\, |A|^x_y \to |B|^{f(x)}_w$$

$$|\forall z^\tau A(z)|^f_{y,c} \equiv \forall z \leq^*_\tau c\, |A(z)|^{f(c)}_z$$

$$|\exists z^\tau A(z)|^{x,c}_y \equiv \exists z \leq^*_\tau c \forall y' \leq^* y\, |A(z)|^x_{y'}$$

$$n \leq^*_{\mathbb{N}} m \equiv n \leq m$$

$$f \leq^*_{\rho\to\tau} g \equiv \forall a^\rho \forall x \leq^*_\rho a (fx \leq^*_\tau ga \wedge gx \leq^*_\tau ga)$$

$$|A \to B|^{f,g}_{x,w} \equiv \forall y \leq^* g(x,w) |A|^x_y \to |B|^{f(x)}_w$$

$$|\forall z^\tau A(z)|^f_{y,c} \equiv \forall z \leq^*_\tau c |A(z)|^{f(c)}_z$$

$$|\exists z^\tau A(z)|^{x,c}_y \equiv \exists z \leq^*_\tau c \forall y' \leq^* y |A(z)|^x_{y'}$$

Negative bounds for a premise become positive bounds

Universal quantifiers are "negatively" bounded

Good news:

Contra Collection:

$$\forall k \exists x \leq^*_\tau a \forall n \leq k A \to \exists x \leq^*_\tau a \forall n A$$

Bad news:

Intensional $\leq^*_\tau$

Axiomitised via "rule"

A functional interpretation for nonstandard arithmetic

Benno van den Berg[a,*,1], Eyvind Briseid[b,2], Pavol Safarik[c,3]

Annals of Pure and Applied Logic 163 (2012) 1962–1994

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \wedge |B|^{v}_{w}$$

$$|A \vee B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \vee |B|^{v}_{w}$$

$$|A \to B|^{f,g}_{x,w} \equiv \forall y \in g[x,w] \, |A|^{x}_{y} \to |B|^{f[x]}_{w}$$

$$|\forall z^{\mathrm{st}_\tau} A(z)|^{f}_{y,z} \equiv |A(z)|^{f[z]}_{y}$$

$$|\exists z^{\mathrm{st}_\tau} A(z)|^{x,z}_{y} \equiv \exists z' \in z \, \forall y' \in y \, |A(z')|^{x}_{y'}$$

$$f[x] := \bigcup_{f' \in f} f'(x)$$

Good news:

Various non-standard principles

Bad news:

Transfer as a "rule"

Functional Interpretations of $\rho \to \tau$

(an alternative approach)

Interpretation of Quantifiers

$$\forall x^{\tau} A \equiv \forall x(\tau(x) \to A)$$
$$\exists x^{\tau} A \equiv \exists x(\tau(x) \wedge A)$$

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \wedge |B|^{v}_{w}$$
$$|A \to B|^{f,g}_{x,w} \equiv \forall y \in g(x,w) |A|^{x}_{y} \to |B|^{f(x)}_{w}$$
$$|\forall z A(z)|^{x}_{y} \equiv \forall z |A(z)|^{x}_{y}$$
$$|\exists z A(z)|^{x}_{y} \equiv \exists z |A(z)|^{x}_{y}$$

Treat quantifiers "uniformly"

$$|\tau(x)|^{a} \equiv x = a$$
$$|\tau(x)|^{a} \equiv x \leq^{*} a$$
$$|\tau(x)|^{a} \equiv x \in a$$

Witness the "qualifiers"
(various alternatives)

$\forall y \sqsubset a \, \lvert A \rvert^x_y$	$x \prec_P a$	**Interpretation**
$\forall y \, \lvert A \rvert^x_y$	$\tau(x) \wedge (x = a)$	Kreisel modified realizability
$\forall y \in a \, \lvert A \rvert^x_y$	$\tau(x) \wedge (x = a)$	Diller-Nahm interpretation
$\lvert A \rvert^x_a$	$\tau(x) \wedge (x = a)$	Gödel's Dialectica interpretation
$\forall y \, \lvert A \rvert^x_y$	$\tau(x) \wedge (x \leq^* a)$	bounded modified realizability
$\forall y \leq^* a \, \lvert A \rvert^x_y$	$\tau(x) \wedge (x \leq^* a)$	bounded functional interpretation
$\forall y \, \lvert A \rvert^x_y$	$\mathrm{st}(x) \wedge (x \in a)$	Herbrand modified realizability
$\forall y \in a \, \lvert A \rvert^x_y$	$\mathrm{st}(x) \wedge (x \in a)$	Herbrand functional interpretation

B. Dinis and P. Oliva, **Parametrised functional interpretations**, in preparation

Proposal

1. Work in first-order logic with equality
2. Assume some basic sorts

$$\begin{array}{lcl} \mathbb{N}(x) & \simeq & x \text{ is a natural number} \\ \mathbb{R}(x) & \simeq & x \text{ is a real} \\ X(x) & \simeq & x \text{ is a point in a Hilbert space} \end{array}$$

3. Function sorts are "defined notions"

$$\begin{array}{lcl} (\mathbb{N} \to \mathbb{R})(f) & \equiv & \forall x(\mathbb{N}(x) \to \mathbb{R}(fx)) \\ (X \to X)(f) & \equiv & \forall x(X(x) \to X(fx)) \end{array}$$

4. Uses application function:

fx is an abbreviation for $\mathrm{App}(f,x)$

$$x = y \to \mathrm{App}(f,x) = \mathrm{App}(f,y)$$

$$\mathbb{N}^{\mathbb{N}}(\alpha) \equiv \forall x(\mathbb{N}(x) \to \mathbb{N}(\alpha x))$$
$$(\mathbb{N}^{\mathbb{N}} \to \mathbb{N})(\phi) \equiv \forall \alpha(\mathbb{N}^{\mathbb{N}}(\alpha) \to \mathbb{N}(\phi\alpha))$$

$$|\mathbb{N}(x)|^{n} \equiv x = n$$

Marvok principle still valid

$$\begin{aligned} |\mathbb{N}^{\mathbb{N}}(\alpha)|_{n}^{f} &\equiv |\forall x(\mathbb{N}(x) \to \mathbb{N}(\alpha x))|_{n}^{f} \\ &\equiv \forall x,n(|\mathbb{N}(x)|^{n} \to |\mathbb{N}(\alpha x)|^{fn}) \\ &\equiv \forall x,n(x = n \to \alpha x = fn) \\ &\equiv \alpha n = fn \end{aligned}$$

Extensionality axiom for $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ becomes interpretable!

$$\begin{aligned} |(\mathbb{N}^{\mathbb{N}} \to \mathbb{N})(\phi)|_{f}^{\omega,\Delta} &\equiv |\forall \alpha(\mathbb{N}^{\mathbb{N}}(\alpha) \to \mathbb{N}(\phi\alpha))|_{f}^{\omega,\Delta} \\ &\equiv \forall \alpha(\forall n \in \omega f\, |\mathbb{N}^{\mathbb{N}}(\alpha)|_{n}^{f} \to |\mathbb{N}(\phi\alpha)|^{\Delta f}) \\ &\equiv \forall \alpha(\forall n \in \omega f\,(\alpha n = fn) \to \phi\alpha = \Delta f) \end{aligned}$$

$$\mathbb{N}^{\mathbb{N}}(\alpha) \equiv \forall x(\mathbb{N}(x) \to \mathbb{N}(\alpha x))$$
$$(\mathbb{N}^{\mathbb{N}} \to \mathbb{N})(\phi) \equiv \forall \alpha(\mathbb{N}^{\mathbb{N}}(\alpha) \to \mathbb{N}(\phi\alpha))$$

$$|\mathbb{N}(x)|^n \equiv x \leq n$$

No need to introduce majorizability relation
It gets "created" by the interpretation

$$|\mathbb{N}^{\mathbb{N}}(\alpha)|_n^f \equiv |\forall x(\mathbb{N}(x) \to \mathbb{N}(\alpha x))|_n^f \equiv \forall x(|\mathbb{N}(x)|^n \to |\mathbb{N}(\alpha x)|^{fn}) \equiv \forall x \leq n(\alpha x \leq fn)$$

$$|(\mathbb{N}^{\mathbb{N}} \to \mathbb{N})(\phi)|_f^{\omega,\Delta} \equiv |\forall \alpha(\mathbb{N}^{\mathbb{N}}(\alpha) \to \mathbb{N}(\phi\alpha))|_f^{\omega,\Delta} \equiv \forall \alpha(\forall n \in \omega f\, |\mathbb{N}^{\mathbb{N}}(\alpha)|_n^f \to |\mathbb{N}(\phi\alpha)|^{\Delta f}) \equiv \forall \alpha(\forall n \in \omega f \forall x \leq n(\alpha x \leq fn) \to \phi\alpha \leq \Delta f)$$

A new "majorizability" notion for higher-types

Abstract Spaces

This seems to work very well with abstract spaces, e.g.

$\lvert\mathbb{N}(x)\rvert^{n}$	$\equiv$	$x \leq n$	
$\lvert\mathbb{R}(x)\rvert^{q_l,q_u}$	$\equiv$	$q_l \leq x \leq q_u$	
$\lvert X(x)\rvert^{n}$	$\equiv$	$d(x,r) \leq n$	(metric spaces, fixed ref. point r)
$\lvert X(x)\rvert^{n}$	$\equiv$	$\lVert x \rVert \leq n$	(normed spaces)

No need to work with "representations"
(representations created by interpretation)

No need to define (or extend) majorizability
(it also gets created by interpretation)

Summary

work in progress...

Work in a standard first-order theory (with equality)

Functional sorts defined from base sorts

Interpretation of base sorts determines functional sorts

No need to work with “representations” (they are created by inter.)

No need to define majorizability (it is also created by inter.)

Issues with “rules” versus “axioms” seem to disappear

P. Oliva, **Unifying functional interpretations**, NDJFL, 47 (2), 2006

G. Ferreira and P. Oliva, **Funct. inter. of intuitionistic linear logic**, CSL, 2009

M.D. Hernest and P. Oliva, **Hybrid functional interpretations**, CiE, 2008

P. Oliva, **Modified realizability interpretation of classical linear logic**, LICS 2007

G. Ferreira and P. Oliva, **Functional interpretations of intuitionistic linear logic**, Logical Methods in Computer Science, 7(1), 2011

J. Gaspar and P. Oliva, **Proof interpretations with truth**, MLQ, 56(6):591-610, 2010

P. Oliva, **Kreisel's modified realizability and recent variants**, to appear

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