Proof Mining

Lecture 3: Proof Interpretations

Paulo Oliva
Queen Mary University of London

Proof Society - Summer School

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Plan

Lecture 1: Incomplete Statements

Lecture 2: Proof Translations

Lecture 3: **Proof Interpretations**

Lecture 1 & 2 Recap

Incomplete statements can be strengthened by bounding or witnessing some of the quantifications

Complete

Goldbach conjecture. Every even interger greather than 2 can be expressed as the sum of two primes

Fermat's last theorem. No three positive integers a,b,c satisfy $a^n + b^n = c^n$, for $n > 2$

Incomplete

There are $a,b \in \mathbb{R}$ such that $a,b \notin \mathbb{Q}$ but $a^b \in \mathbb{Q}$

$\sqrt{2}$ is irrational

The set of primes is unbounded

Edelstein f.-point theorem. Any contractive $f : [0,1] \to [0,1]$ has at most one fixed point

Brouwer f.-point theorem. Any continuous $f : [0,1] \to [0,1]$ has a fixed point

$$\boxed{\vdash A \vee \neg A}$$

$$\dfrac{\dfrac{\dfrac{[A]_{\alpha}}{A \vee \neg A} \qquad [\neg(A \vee \neg A)]_{\gamma}}{\dfrac{\bot}{\neg A}\,\alpha}}{\dfrac{\bot}{A \vee \neg A}\,\text{PBC},\gamma}$$

$$\boxed{\vdash \neg\neg(A \vee \neg A)}$$

$$\dfrac{\dfrac{\dfrac{[A]_{\alpha}}{A \vee \neg A} \qquad [\neg(A \vee \neg A)]_{\gamma}}{\dfrac{\bot}{\neg A}\,\alpha}}{\dfrac{\bot}{\neg\neg(A \vee \neg A)}\,\to\text{I},\gamma}$$

Intuitionistic Logic
(ex falso quodlibet)

$$\dfrac{\dfrac{\vdots}{\bot}}{A}\ \text{EFQ}$$

Classical Logic
(proof by contraction)

$$\dfrac{\dfrac{\dfrac{[\neg A]_{\alpha}}{\vdots}}{\bot}}{A}\ \text{PBC},\ \alpha$$

Proposition (Gentzen 1933).

If CL proves $\Gamma \vdash A$ then IL proves $\Gamma^N \vdash A^N$ where

$$\begin{array}{lcllcl}
(A \wedge B)^N & \equiv & A^N \wedge B^N & (P)^* & \equiv & \neg\neg P \\
(A \vee B)^N & \equiv & \neg\neg(A^N \vee B^N) & (\forall x\, A)^N & \equiv & \forall x\, A^N \\
(A \to B)^N & \equiv & A^N \to B^N & (\exists x\, A)^N & \equiv & \neg\neg \exists x A^N
\end{array}$$

Comprehension

$$\exists f^{\mathbb{N}\to\mathbb{B}}\forall n^{\mathbb{N}}(fn \leftrightarrow A(n))$$

Let $T(n,n,u)$ be the statement that Turing machine with code n on input n will halt with computation u

The Halting problem (known to be undecidable) is

$$A(n) \equiv \exists u T(n,n,u)$$

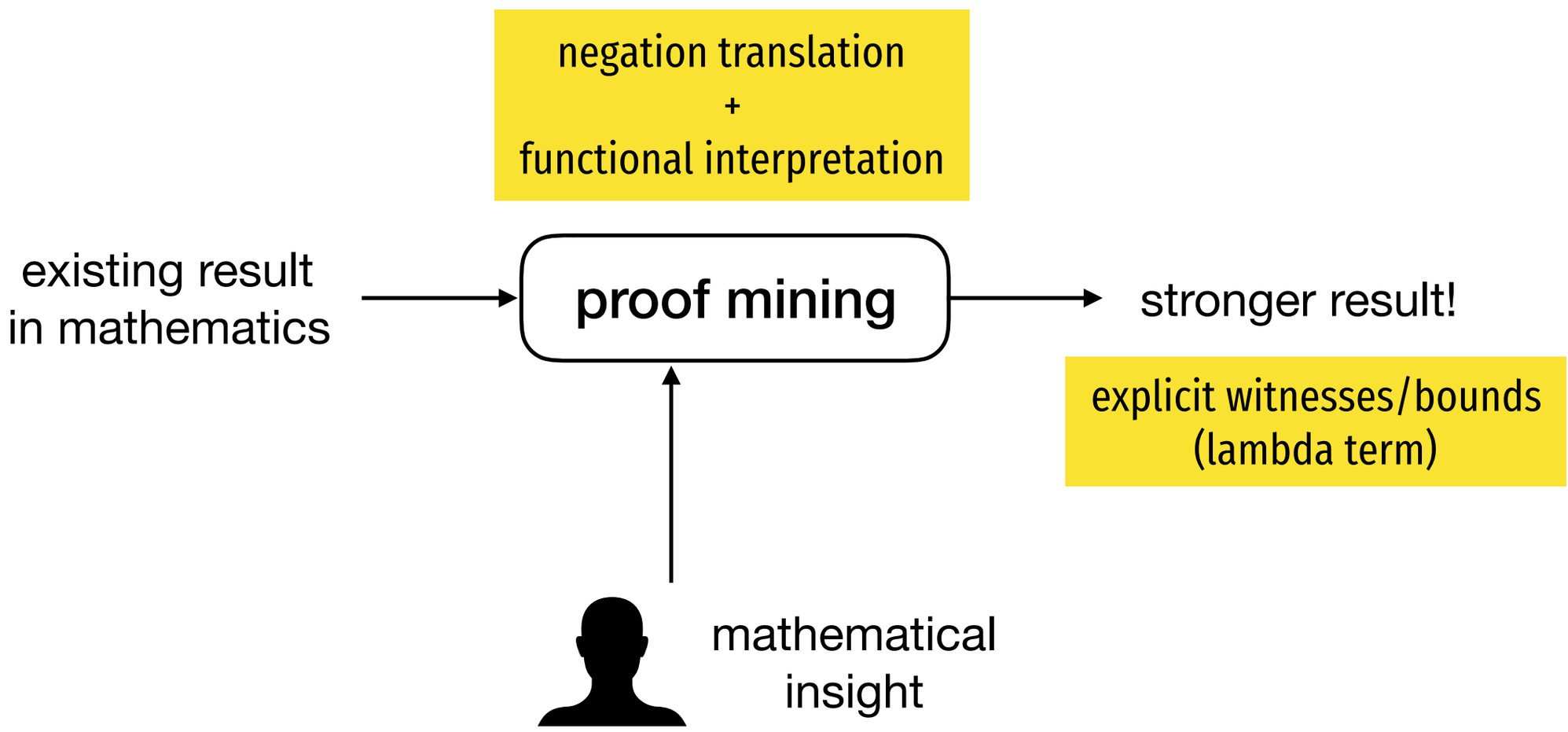
So clearly, the following (true) statement cannot be witnessed by a computable function f

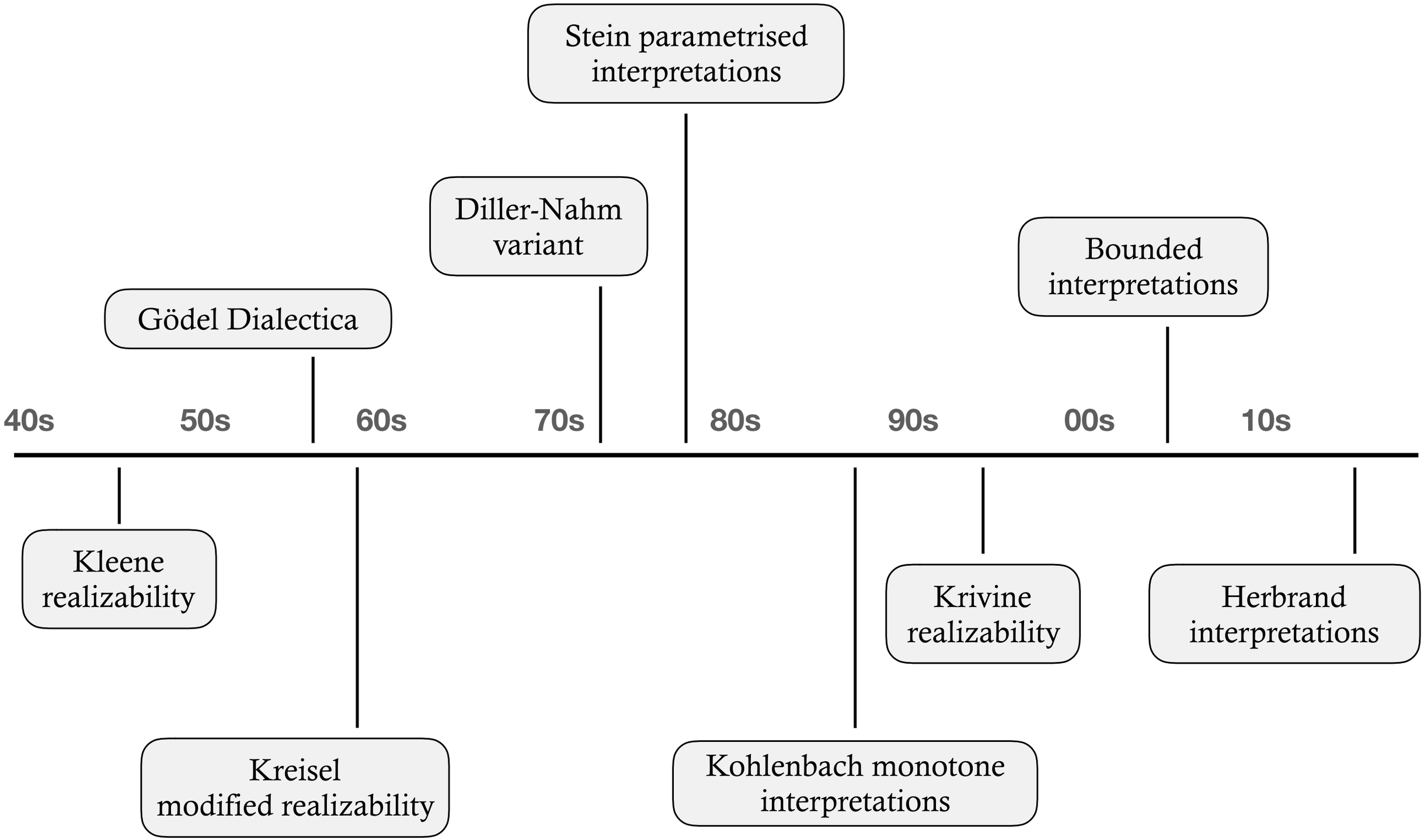
$$\exists f^{\mathbb{N}\to\mathbb{B}}\forall n^{\mathbb{N}}(fn \leftrightarrow \exists u T(n,n,u))$$

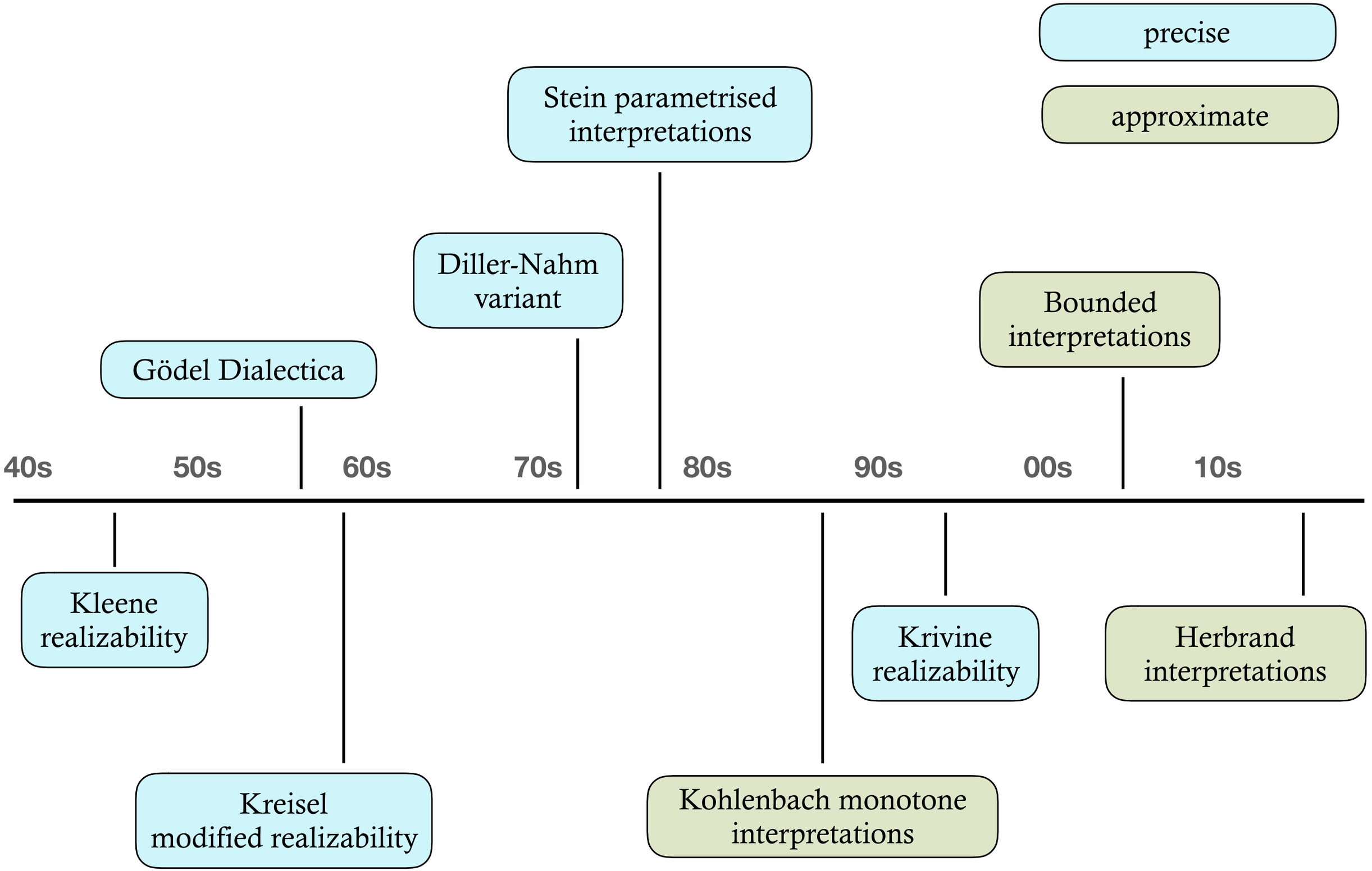
Today

- Functional interpretations
- Lambda calculus, system T
- Interpreting induction
- Interpreting choice and comprehension

Proof Mining




Stein parametrised interpretations
Diller-Nahm variant
Gödel Dialectica
Bounded interpretations
40s
50s
60s
70s
80s
90s
00s
10s
Kleene realizability
Kreisel modified realizability
Kohlenbach monotone interpretations
Krivine realizability
Herbrand interpretations


precise
approximate
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Functional Interpretations

Lemma 1: $\forall p,q>0\left(\neg\text{Even}(p)\vee\neg\text{Even}(q)\to p^2\neq 2q^2\right)$

Lemma 2: $\forall x,y(x^2\neq y^2\to x\neq y)$

Theorem: $\forall p,q>0\left(\neg\text{Even}(p)\vee\neg\text{Even}(q)\to\frac{p}{q}\neq\sqrt{2}\right)$

Lemma 1^*: $\forall p,q>0\left(\neg\text{Even}(p)\vee\neg\text{Even}(q)\to|p^2-2q^2|\geq 1\right)$

Lemma 2^*: $\forall x,y,\delta(|x^2-y^2|\geq\delta\to|x-y|\geq\frac{\delta}{x+y})$

Theorem *: $\forall p,q>0\left(\neg\text{Even}(p)\vee\neg\text{Even}(q)\to|\frac{p}{q}-\sqrt{2}|\geq\frac{1}{pq+2q^2}\right)$

Lemma 1. $\exists x A(x) \to \exists y B(y)$ $\longrightarrow$ $\forall x(A(x) \to B(gx))$

Lemma 2. $\exists y B(y) \to \exists z C(z)$ $\longrightarrow$ $\forall y(B(y) \to C(hy))$

Theorem. $\exists x A(x) \to \exists z C(z)$

$$\dfrac{\dfrac{\exists x A(x) \quad \exists x A(x) \to \exists y B(y)}{\exists y B(y)} \qquad \exists y B(y) \to \exists z C(z)}{\exists z C(z)}$$

$A(x)$ $\longrightarrow$ $C(fx)$

f ?

Lemma 1. $\exists x A(x) \to \exists y B(y) \longrightarrow \forall x(A(x) \to B(gx))$

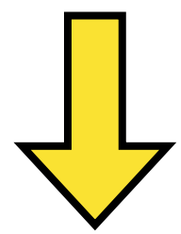
Lemma 2. $\exists y B(y) \to \exists z C(z) \longrightarrow \forall y(B(y) \to C(hy))$

Theorem. $\exists x A(x) \to \exists z C(z) \longrightarrow \forall x(A(x) \to C(h(gx)))$

$$\dfrac{\dfrac{[A(v)]_\alpha \quad \dfrac{\forall x(A(x) \to B(gx))}{A(v) \to B(gv)}}{B(gv)} \quad \dfrac{\forall y(B(y) \to C(hy))}{B(gv) \to C(h(gv))}}{\dfrac{\dfrac{C(h(gv))}{A(v) \to C(h(gv))}\,\alpha}{\forall x(A(x) \to C(h(gx)))}}$$

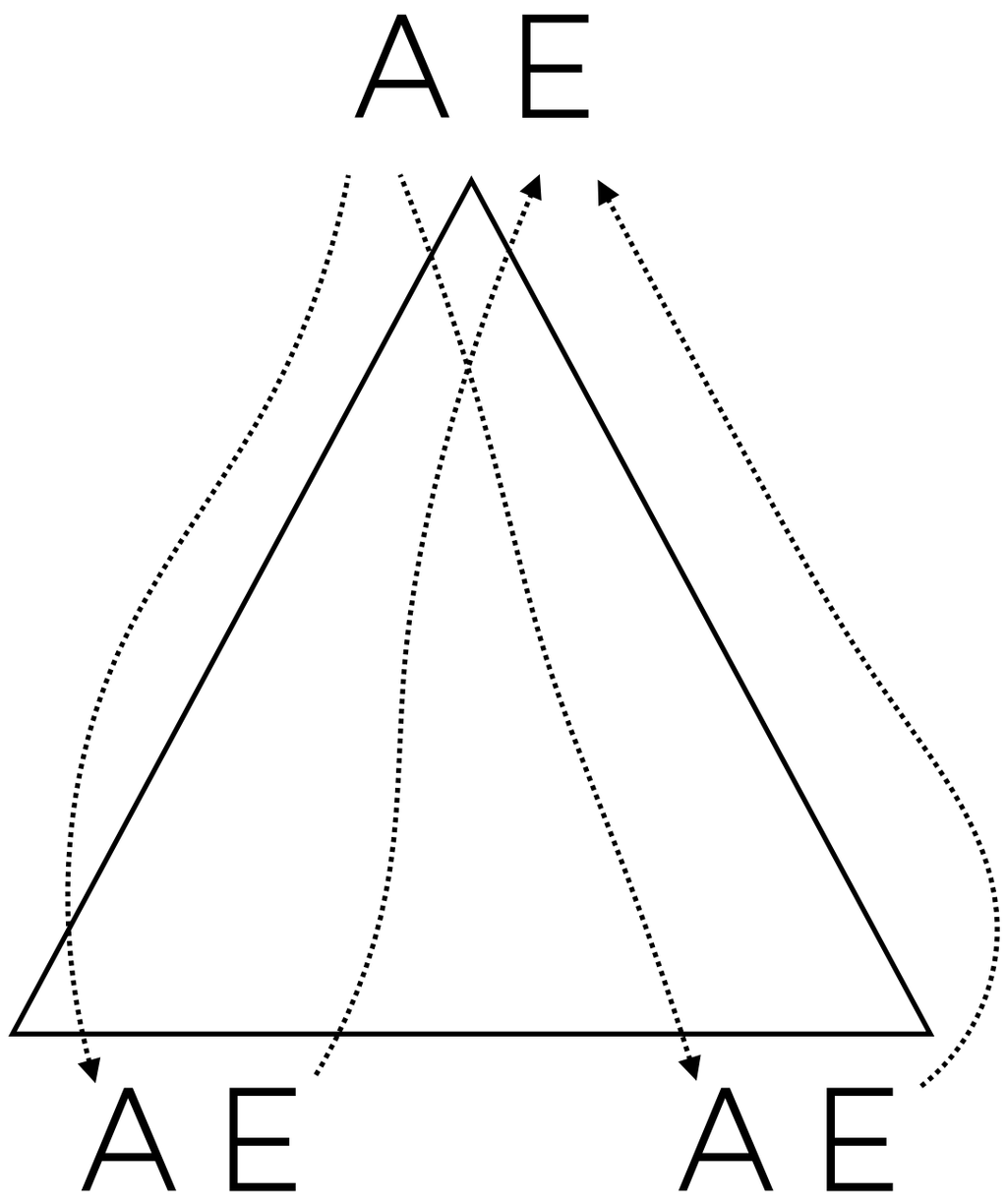
$$\dfrac{\dfrac{\exists x A(x) \quad \exists x A(x) \to \exists y B(y)}{\exists y B(y)} \qquad \exists y B(y) \to \exists z C(z)}{\exists z C(z)}$$

functional interpretations perform such proof transformations



$$\dfrac{[A(v)]_\alpha \quad \dfrac{\forall x(A(x) \to B(gx))}{A(v) \to B(gv)}}{\dfrac{B(gv) \qquad B(gv) \to C(h(gv))}{\dfrac{C(h(gv))}{A(v) \to C(h(gv))}\,\alpha}} \qquad \dfrac{\forall y(B(y) \to C(hy))}{B(gv) \to C(h(gv))}$$

$$\forall x(A(x) \to C(h(gx)))$$


A E
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Lemma 1. $\forall x A(x) \to \forall y B(y)$ $\longrightarrow$ $\forall y(A(gy) \to B(y))$

Lemma 2. $\forall y B(y) \to \forall z C(z)$ $\longrightarrow$ $\forall z(B(hz) \to C(z))$

Theorem. $\forall x A(x) \to \forall z C(z)$

$$\dfrac{\dfrac{\forall x A(x) \quad \forall x A(x) \to \forall y B(y)}{\forall y B(y)} \qquad \forall y B(y) \to \forall z C(z)}{\forall z C(z)}$$

$A(fz)$

$C(z)$

$f\,?$

Lemma 1. $\forall x A(x) \to \forall y B(y)$ $\longrightarrow$ $\forall y(A(gy) \to B(y))$

Lemma 2. $\forall y B(y) \to \forall z C(z)$ $\longrightarrow$ $\forall z(B(hz) \to C(z))$

Theorem. $\forall x A(x) \to \forall z C(z)$ $\longrightarrow$ $\forall z(A(g(hz)) \to C(z))$

$$\dfrac{\dfrac{[A(g(hz)]_{\alpha} \quad \dfrac{\forall y(A(gy) \to B(y))}{A(g(hz)) \to B(hz)}}{B(hz)} \qquad \dfrac{\forall z(B(hz) \to C(z))}{B(hz) \to C(z)}}{\dfrac{\dfrac{C(z)}{A(g(hz)) \to C(z)}\,\alpha}{\forall z(A(g(hz)) \to C(z))}}$$

Gödel's Dialectica Interpretation

ÜBER EINE BISHER NOCH NICHT BENÜTZTE ERWEITERUNG DES FINITEN STANDPUNKTES

von Kurt Gödel, Princeton

Dialectica, vol. 12, 1958

$$
\begin{aligned}
|A \wedge B|^{x,v}_{y,w} &\equiv |A|^{x}_{y} \wedge |B|^{v}_{w} \\
|A \vee B|^{x,v,b}_{y,w} &\equiv (b{=}0 \wedge |A|^{x}_{y}) \vee (b \neq 0 \wedge |B|^{v}_{w}) \\
|A \to B|^{f,g}_{x,w} &\equiv |A|^{x}_{g(x,w)} \to |B|^{f(x)}_{w} \\
|\forall z\, A(z)|^{f}_{y,z} &\equiv |A(z)|^{f(z)}_{y} \\
|\exists z\, A(z)|^{x,z}_{y} &\equiv |A(z)|^{x}_{y}
\end{aligned}
$$

Theorem (Gödel'58). If HA $\vdash A$ then there exists a term t of system T such that T $\vdash |A|^{t}_{y}$

$$
\begin{aligned}
|A \wedge B|^{x,v}_{y,w} &\equiv |A|^{x}_{y} \wedge |B|^{v}_{w} \\
|A \vee B|^{x,v,b}_{y,w} &\equiv (b{=}0 \wedge |A|^{x}_{y}) \vee (b \neq 0 \wedge |B|^{v}_{w}) \\
|A \to B|^{f,g}_{x,w} &\equiv |A|^{x}_{g(x,w)} \to |B|^{f(x)}_{w} \\
|\forall z\, A(z)|^{f}_{y,z} &\equiv |A(z)|^{f(z)}_{y} \\
|\exists z\, A(z)|^{x,z}_{y} &\equiv |A(z)|^{x}_{y}
\end{aligned}
$$

$$A = \sqrt{2} \notin \mathbb{Q} \qquad |A|^{f}_{p,q} \equiv |\frac{p}{q} - \sqrt{2}| > f(p,q)$$

$$A = \forall n \exists p \in \mathbb{P}(p > n) \qquad |A|^{f}_{n} \equiv fn \in \mathbb{P} \wedge fn \geq n$$

$$\boxed{\forall n(A(n)\to A(n+1))\vdash A(0)\to A(2)}$$

assumption used twice (contraction)

$$\dfrac{[A(0)]_{\alpha}\qquad \dfrac{\forall n(A(n)\to A(n+1))}{A(0)\to A(1)}}{A(1)}\qquad \dfrac{\forall n(A(n)\to A(n+1))}{A(1)\to A(2)}$$

$$A(2)$$

Is there an a single a such that:

$$\boxed{A(a)\to A(a+1)\vdash A(0)\to A(2)}$$

Is there an a single a such that:

$$\boxed{A(a) \to A(a+1) \vdash A(0) \to A(2)}$$

In fact there is, when the formulas are “decidable”

$$a := \left\{ \begin{array}{ll} 0 & \text{if } \neg(A(0) \to A(1)) \\ 1 & \text{otherwise} \end{array} \right\}$$

Works, but… a bit of a hack

Diller-Nahm Variant

Diller-Nahm Interpretation

$$
\begin{aligned}
|A \wedge B|^{x,v}_{y,w} &\equiv |A|^{x}_{y} \wedge |B|^{v}_{w} \\
|A \vee B|^{x,v,b}_{y,w} &\equiv (b{=}0 \wedge |A|^{x}_{y}) \vee (b \neq 0 \wedge |B|^{v}_{w}) \\
|A \to B|^{f,g}_{x,w} &\equiv \forall y \in g(x,w)\, |A|^{x}_{y} \to |B|^{f(x)}_{w} \\
|\forall z\, A(z)|^{f}_{y,z} &\equiv |A(z)|^{f(z)}_{y} \\
|\exists z\, A(z)|^{x,z}_{y} &\equiv |A(z)|^{x}_{y}
\end{aligned}
$$

precise witnesses for positive existential,
approximating set for negative universals

Is there an a finite set S such that:

$$\boxed{\forall a \in S(A(a) \to A(a+1)) \vdash A(0) \to A(2)}$$

Yes!

$$\boxed{S := \{0,1\}}$$

Indeed:

$$\boxed{\forall a \in \{0,1\}(A(a) \to A(a+1)) \vdash A(0) \to A(2)}$$

Interpreting Induction

$$\frac{\exists k A(0,k) \quad \forall n^{\mathbb{N}}(\exists k A(n,k) \to \exists k A(n+1,k))}{\forall n^{\mathbb{N}} \exists k A(n,k)}$$

By induction hypothesis we have:

$$A(0,k_0) \quad \text{and} \quad \forall n^{\mathbb{N}}(A(n,k) \to A(n+1, fnk))$$

So witness for conclusion is:

$$k := fn(f(n-1)(\ldots(f\,0k_0)) = \mathrm{Rec}\, n\, k_0\, f$$

$$\mathrm{Rec}\ 0\ k_0\ f\ =\ k_0$$

$$\mathrm{Rec}\ (n+1)\ k_0\ f = f\ n\ (\mathrm{Rec}\ n\ k_0\ f)$$

Gödel system T
=
lambda calculus + Rec

Choice, Compactness, etc…

Comprehension

$$\exists f^{\mathbb{N}\to\mathbb{B}}\forall n^{\mathbb{N}}(fn \leftrightarrow A(n))$$

Let $T(n,n,u)$ be the statement that Turing machine with code n on input n will halt with computation u

The Halting problem (known to be undecidable) is

$$A(n) \equiv \exists u T(n,n,u)$$

So clearly, the following (true) statement cannot be witnessed by a computable function f

$$\exists f^{\mathbb{N}\to\mathbb{B}}\forall n^{\mathbb{N}}(fn \leftrightarrow \exists u T(n,n,u))$$

Comprehension

$$\exists f^{\mathbb{N}\to\mathbb{B}}\forall n^{\mathbb{N}}(fn \leftrightarrow A(n))$$

Approach I

1. Apply double negation translation

$$\neg\neg\exists f^{\mathbb{N}\to\mathbb{B}}\forall n^{\mathbb{N}}(fn \leftrightarrow A^{N}(n))$$

2. Extend system T so as to be able to witness this

   E.g. system T + recursion on well-founded trees

   (bar recursion)

"Life offers a cruel choice: you can be right or happy. Not both."

- Albert J. Bernstein

Precise

$$f(x)=\left\{\begin{array}{ll} 0 & \text{if } \exists u.T(x,x,u) \\ 1 & \text{if } \forall u.\neg T(x,x,u) \end{array}\right\}$$

precise

but

non-computable

Approximate

$$f(x)=\{0,\ 1\}$$

approximate

but

computable

Comprehension

$$\exists f^{\mathbb{N}\to\mathbb{B}}\forall n^{\mathbb{N}}(fn \leftrightarrow A(n))$$

Approach II

1. Notice that f can be easily "bounded"

$$\exists f^{\mathbb{N}\to\mathbb{B}} \leq 1\forall n^{\mathbb{N}}(fn \leftrightarrow A^{N}(n))$$

2. Work with "approximate" functional interpretations

   E.g. Monotone, bounded, Herbrand, … interpretations

Lemma 1. $\exists x A(x) \to \exists y B(y)$ $\longrightarrow$ $\exists x \leq n\, A(x) \to \exists y \leq gn B(y)$

Lemma 2. $\exists y B(y) \to \exists z C(z)$ $\longrightarrow$ $\exists y \leq n B(y) \to \exists z \leq hn C(z)$

Theorem. $\exists x A(x) \to \exists z C(z)$ $\longrightarrow$ $\exists x \leq n\, A(x) \to \exists z \leq h(gn)\, C(z)$

$\exists x \leq n\, A(x)$

$$\frac{\exists x A(x) \quad \exists x A(x) \to \exists y B(y)}{\exists y B(y)} \qquad \frac{\exists y B(y) \to \exists z C(z)}{\exists z C(z)}$$

f ? : $\exists z \leq fn\ C(z)$

Lemma 1. $\exists x A(x) \to \exists y B(y)$ $\longrightarrow$ $\exists x \leq n\, A(x) \to \exists y \leq gn B(y)$

Lemma 2. $\exists y B(y) \to \exists z C(z)$ $\longrightarrow$ $\exists y \leq n B(y) \to \exists z \leq hn C(z)$

Theorem. $\exists x A(x) \to \exists z C(z)$ $\longrightarrow$ $\exists x \leq n\, A(x) \to \exists z \leq h(gn) C(z)$

$$\dfrac{\dfrac{\exists x \leq n A(x) \quad \exists x \leq n A(x) \to \exists y \leq gn B(y)}{\exists y \leq gn B(y)} \qquad \dfrac{\exists y \leq n B(y) \to \exists z \leq hn C(z)}{\exists y \leq gn B(y) \to \exists z \leq h(gn) C(z)}}{\exists z \leq h(gn) C(z)}$$

Approximate Interpretation

$$
\begin{aligned}
|A \wedge B|^{x,v}_{y,w} &\equiv |A|^{x}_{y} \wedge |B|^{v}_{w} \\
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|A \to B|^{f,g}_{x,w} &\equiv \forall y \in g(x,w)\, |A|^{x}_{y} \to |B|^{f(x)}_{w} \\
|\forall z^{\mathbb{N}} A(z)|^{f}_{y,a} &\equiv \forall z \leq a\, |A(z)|^{f(a)}_{y} \\
|\exists z^{\mathbb{N}} A(z)|^{x,a}_{y} &\equiv \exists z \leq a\, |A(z)|^{x}_{y}
\end{aligned}
$$

P. Oliva, **Unifying functional interpretations**, NDJFL, 47 (2), 2006

G. Ferreira and P. Oliva, **Funct. inter. of intuitionistic linear logic**, CSL, 2009

M.D. Hernest and P. Oliva, **Hybrid functional interpretations**, CiE, 2008

P. Oliva, **Modified realizability interpretation of classical linear logic**, LICS 2007

G. Ferreira and P. Oliva, **Functional interpretations of intuitionistic linear logic**, Logical Methods in Computer Science, 7(1), 2011

J. Gaspar and P. Oliva, **Proof interpretations with truth**, MLQ, 56(6):591-610, 2010

P. Oliva, **Kreisel's modified realizability and recent variants**, to appear

B. Dinis and P. Oliva, **Parametrised functional interpretations**, in preparation