Relational Proof Interpretations

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Logic Colloquium

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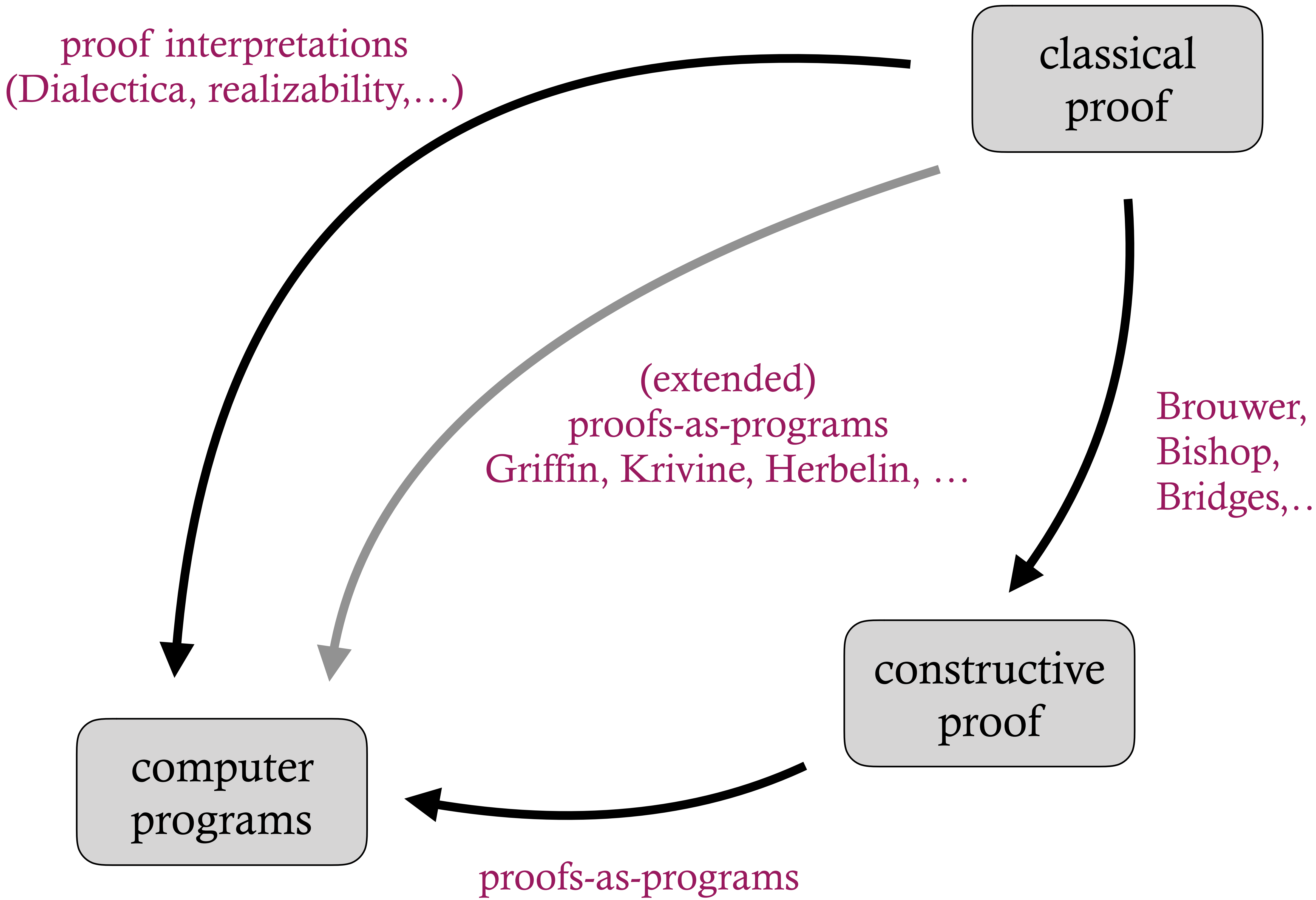

proof interpretations
(Dialectica, realizability,…)
classical
proof
(extended)
proofs-as-programs
Griffin, Krivine, Herbelin, …
Brouwer,
Bishop,
Bridges,…
constructive
proof
computer
programs
proofs-as-programs

Plan

- **Part 1: Sets vs Relations**
  (*intuitionistic logic*)

relational approach
is more general

- **Part 2: Unification**
  (*linear logic*)

interpretations (only)
differ in treatment of $!A$

- **Part 3: Games and Applications**
  (*classical logic*)

higher-order games explain
higher-order programs

Part 1:

Sets vs Relations

(*realizability vs dialectica*)

THE JOURNAL OF SYMBOLIC LOGIC
Volume 10, Number 4, Dec. 1945

ON THE INTERPRETATION OF INTUITIONISTIC NUMBER THEORY

S. C. KLEENE

$$
\begin{array}{lll}
n \mathrel{\mathrm{r}} (s = t) & \equiv & (n = 0) \wedge (s = t) \\
n \mathrel{\mathrm{r}} A \wedge B & \equiv & n_0 \mathrel{\mathrm{r}} A \wedge n_1 \mathrel{\mathrm{r}} B \\
n \mathrel{\mathrm{r}} A \vee B & \equiv & (n_0 = 0 \wedge n_1 \mathrel{\mathrm{r}} A) \vee (n_0 \neq 1 \wedge n_1 \mathrel{\mathrm{r}} B) \\
n \mathrel{\mathrm{r}} A \rightarrow B & \equiv & \forall a (a \mathrel{\mathrm{r}} A \rightarrow \{n\}(a)\downarrow \wedge \{n\}(a) \mathrel{\mathrm{r}} B) \\
n \mathrel{\mathrm{r}} \exists z A(z) & \equiv & n_1 \mathrel{\mathrm{r}} A(n_0) \\
n \mathrel{\mathrm{r}} \forall z A(z) & \equiv & \forall x (\{n\}(x)\downarrow \wedge \{n\}(x) \mathrel{\mathrm{r}} A(x))
\end{array}
$$

Theorem (Kleene-Nelson).

If HA $\vdash A$ then HA $\vdash n \mathrel{\mathrm{r}} A$, for some numeral n

Realizability

$$\underbrace{A}_{\text{sentence}} \quad \mapsto \quad \underbrace{\{\, n : n \mathrel{\mathrm{r}} A \,\}}_{\text{set of realizers of } A}$$

ÜBER EINE BISHER NOCH NICHT BENÜTZTE ERWEITERUNG DES FINITEN STANDPUNKTES

von Kurt GÖDEL, Princeton

Dialectica, vol. 12, 1958

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^{x}_{y} \wedge |B|^{v}_{w}$$

$$|A \vee B|^{x,v,b}_{y,w} \equiv (b=0 \wedge |A|^{x}_{y}) \vee (b \neq 1 \wedge |B|^{v}_{w})$$

$$|A \rightarrow B|^{f,g}_{x,w} \equiv |A|^{x}_{g(x,w)} \rightarrow |B|^{f(x)}_{w}$$

$$|\forall z\, A(z)|^{f}_{y,s} \equiv |A(s)|^{f(s)}_{y}$$

$$|\exists z\, A(z)|^{x,s}_{y} \equiv |A(s)|^{x}_{y}$$

Theorem (Gödel).

If HA $\vdash$ A then T $\vdash$ $\forall y\, |A|^{t}_{y}$ for some term $t \in$ T

Dialectica Interpretation

$$\underbrace{A}_{\text{sentence}} \quad \mapsto \quad \{\, ({\color{green}x}, {\color{red}y}) : \underbrace{|A|^{\color{green}x}_{\color{red}y}}_{\text{relation between arguments and counter-arguments}} \,\}$$

sentence

relation between arguments

and counter-arguments

Example

α is eventually bounded $\qquad$ α is bounded

$$A \equiv \exists n \forall i \geq n (\alpha(i) \leq n) \rightarrow \exists k \forall j (\alpha(j) \leq k)$$

$$|A|_{n,j}^{f,g} \equiv (g(n,j) \geq n \rightarrow \alpha(g(n,j)) \leq n) \rightarrow \alpha(j) \leq f(n)$$

$$f(n) = \max\{n, \max\{\alpha(i) \mid i < n\}\}$$

$$g(n,j) = j$$

so… which one is better, sets or relations?

relational approach
is more general

Realizability can also be presented in a 'relational' style

Kleene realizability

$$n \mathrm{\ r\ } A \wedge B \equiv n_0 \mathrm{\ r\ } A \wedge n_1 \mathrm{\ r\ } B$$

$$n \mathrm{\ r\ } A \vee B \equiv (n_0 = 0 \wedge n_1 \mathrm{\ r\ } A) \vee (n_0 \neq 1 \wedge n_1 \mathrm{\ r\ } B)$$

$$n \mathrm{\ r\ } A \rightarrow B \equiv \forall a (a \mathrm{\ r\ } A \rightarrow \{n\}(a)\downarrow \wedge \{n\}(a) \mathrm{\ r\ } B)$$

$$n \mathrm{\ r\ } \exists z A(z) \equiv n_1 \mathrm{\ r\ } A(n_0)$$

$$n \mathrm{\ r\ } \forall z A(z) \equiv \forall x (\{n\}(x)\downarrow \wedge \{n\}(x) \mathrm{\ r\ } A(x))$$

Relational presentation

$$|A \wedge B|_a^n \equiv |A|_{a_0}^{n_0} \wedge |B|_{a_1}^{n_1}$$

$$|A \vee B|_a^n \equiv (n_0 = 0 \wedge |A|_a^{n_1}) \vee (n_0 \neq 1 \wedge |B|_a^{n_1})$$

$$|A \rightarrow B|_a^n \equiv \forall b |A|_b^{a_0} \rightarrow (\{n\}(a_0)\downarrow \wedge |B|_{a_1}^{\{n\}(a_0)})$$

$$|\exists z A(z)|_a^n \equiv |A(n_0)|_a^{n_1}$$

$$|\forall z A(z)|_a^n \equiv \{n\}(a_0)\downarrow \wedge |A(a_0)|_{a_1}^{\{n\}(a_0)}$$

$n \mathrm{\ r\ } A$

iff

$\forall a |A|_a^n$

Kreisel modified realizability

$x, v \text{ mr } A \wedge B$	$\equiv$	$x \text{ mr } A \wedge v \text{ mr } B$
$x, v, b \text{ mr } A \vee B$	$\equiv$	$(b{=}0 \wedge x \text{ mr } A) \vee (b \neq 0 \wedge v \text{ mr } B)$
$f \text{ mr } A \rightarrow B$	$\equiv$	$\forall x(x \text{ mr } A \rightarrow f(x) \text{ mr } B)$
$x, s \text{ mr } \exists z A(z)$	$\equiv$	$x \text{ mr } A(s)$
$f \text{ mr } \forall z A(z)$	$\equiv$	$\forall x(f(x) \text{ mr } A(x))$

Relational presentation

$\lvert A \wedge B \rvert^{x,v}_{y,w}$	$\equiv$	$\lvert A \rvert^{x}_{y} \wedge \lvert B \rvert^{v}_{w}$
$\lvert A \vee B \rvert^{x,v,b}_{y,w}$	$\equiv$	$(b{=}0 \wedge \lvert A \rvert^{x}_{y}) \vee (b \neq 1 \wedge \lvert B \rvert^{v}_{w})$
$\lvert A \rightarrow B \rvert^{f}_{x,w}$	$\equiv$	$\forall y \lvert A \rvert^{x}_{y} \rightarrow \lvert B \rvert^{f(x)}_{w}$
$\lvert \forall z A(z) \rvert^{f}_{y,s}$	$\equiv$	$\lvert A(s) \rvert^{f(s)}_{y}$
$\lvert \exists z A(z) \rvert^{x,s}_{y}$	$\equiv$	$\lvert A(s) \rvert^{x}_{y}$

$x \text{ mr } A$

iff

$\forall y \lvert A \rvert^{x}_{y}$

Gödel Dialectica interpretation

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^x_y \wedge |B|^v_w$$

$$|A \vee B|^{x,v,b}_{y,w} \equiv (b=0 \wedge |A|^x_y) \vee (b \neq 1 \wedge |B|^v_w)$$

$$|A \rightarrow B|^{f,g}_{x,w} \equiv |A|^x_{g(x,w)} \rightarrow |B|^{f(x)}_w$$

$$|\forall z\, A(z)|^f_{y,s} \equiv |A(s)|^{f(s)}_y$$

$$|\exists z\, A(z)|^{x,s}_y \equiv |A(s)|^x_y$$

$$A^D(x,y) \text{ iff } |A|^x_y$$

Kreisel modified realizability

$$|A \wedge B|^{x,v}_{y,w} \equiv |A|^x_y \wedge |B|^v_w$$

$$|A \vee B|^{x,v,b}_{y,w} \equiv (b=0 \wedge |A|^x_y) \vee (b \neq 1 \wedge |B|^v_w)$$

$$|A \rightarrow B|^f_{x,w} \equiv \forall y |A|^x_y \rightarrow |B|^{f(x)}_w$$

$$|\forall z\, A(z)|^f_{y,s} \equiv |A(s)|^{f(s)}_y$$

$$|\exists z\, A(z)|^{x,s}_y \equiv |A(s)|^x_y$$

$$x \text{ mr } A \text{ iff } \forall y |A|^x_y$$

P. Oliva, **Unifying functional interpretations**, NDJFL, 47 (2), 2006

Part 2:

Linear Logic

(*it's all about the bang!*)

Linear Logic

A refinement of classical and intuitionistic logic

$$A \to B \quad \Rightarrow \quad !A \multimap B$$

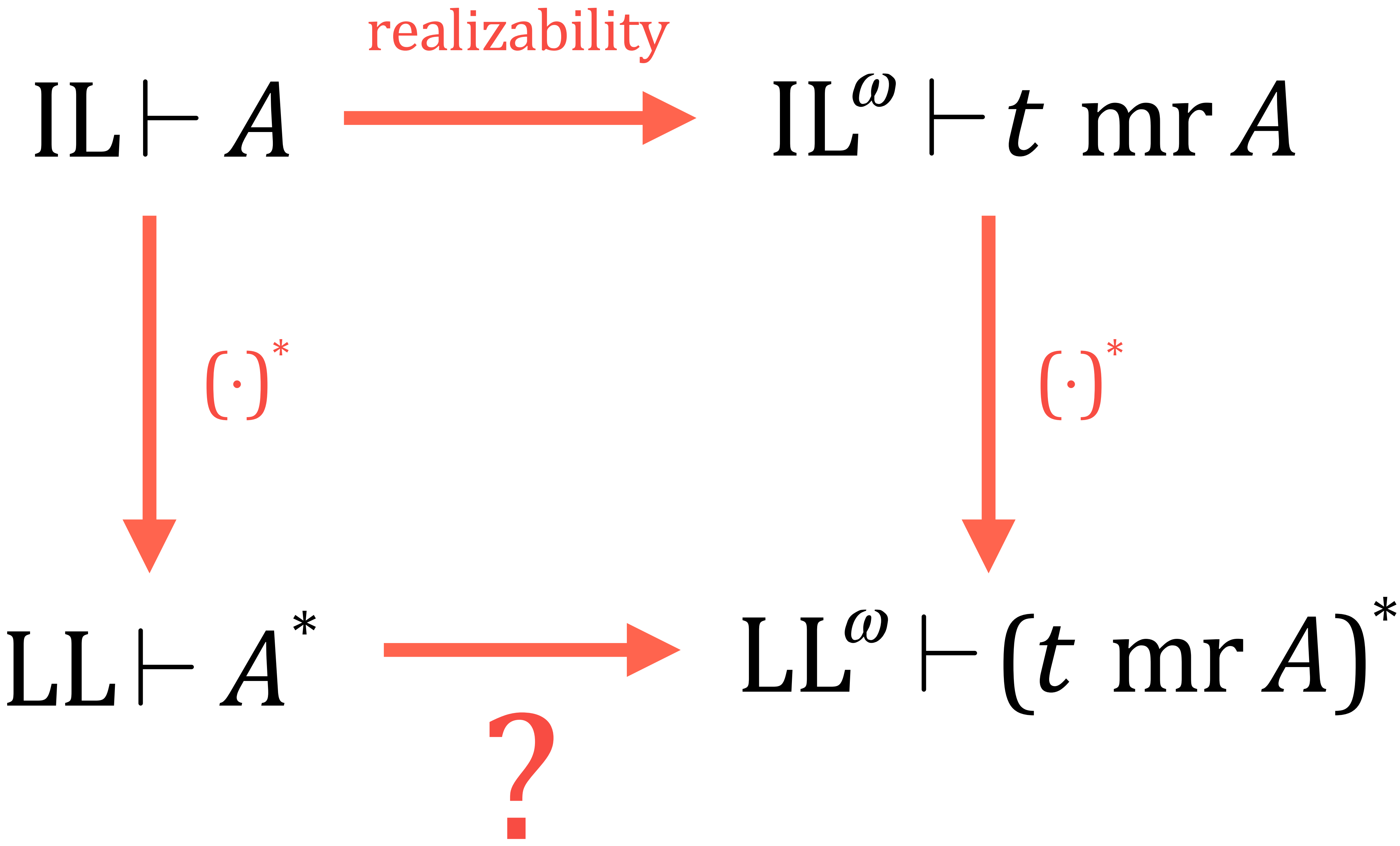
$$A \wedge B \quad \Rightarrow \quad A \,\&\, B$$

$$A \wedge B \quad \Rightarrow \quad A \otimes B$$

call-by-name translation		
$(A \wedge B)^{*}$	$\equiv$	$A^{*} \,\&\, B^{*}$
$(A \vee B)^{*}$	$\equiv$	$!A^{*} \oplus !B^{*}$
$(A \rightarrow B)^{*}$	$\equiv$	$!A^{*} \multimap B^{*}$
$(\forall z\, A)^{*}$	$\equiv$	$\forall z\, A^{*}$
$(\exists z\, A)^{*}$	$\equiv$	$\exists z !A^{*}$

call-by-value translation		
$(A \wedge B)^{\circ}$	$\equiv$	$A^{\circ} \otimes B^{\circ}$
$(A \vee B)^{\circ}$	$\equiv$	$A^{\circ} \oplus B^{\circ}$
$(A \rightarrow B)^{\circ}$	$\equiv$	$!(A^{\circ} \multimap B^{\circ})$
$(\forall z\, A)^{\circ}$	$\equiv$	$!\forall z\, A^{\circ}$
$(\exists z\, A)^{\circ}$	$\equiv$	$\exists z\, A^{\circ}$

$$\mathrm{IL} \vdash A \quad \Longrightarrow \quad \begin{array}{l} \mathrm{LL} \vdash A^{\circ} \\ \mathrm{LL} \vdash A^{*} \end{array}$$


realizability
IL ⊢ A
IL^ω ⊢ t mr A
(·)*
(·)*
LL ⊢ A*
LL^ω ⊢ (t mr A)*
?

Interpretation of Linear Logic

$$|A \otimes B|^{x,v}_{y,w} \equiv |A|^x_y \otimes |B|^v_w$$

$$|A \oplus B|^{x,v,b}_{y,w} \equiv (b{=}0 \,\&\, |A|^x_y) \oplus (b{\neq}1 \,\&\, |B|^v_w)$$

$$|A \,\&\, B|^{x,v}_{y,w,b} \equiv (b{=}0 \,\&\, |A|^x_y) \oplus (b{\neq}1 \,\&\, |B|^v_w)$$

$$|A \multimap B|^{f,g}_{x,w} \equiv |A|^x_{g(x,w)} \multimap |B|^{f(x)}_w$$

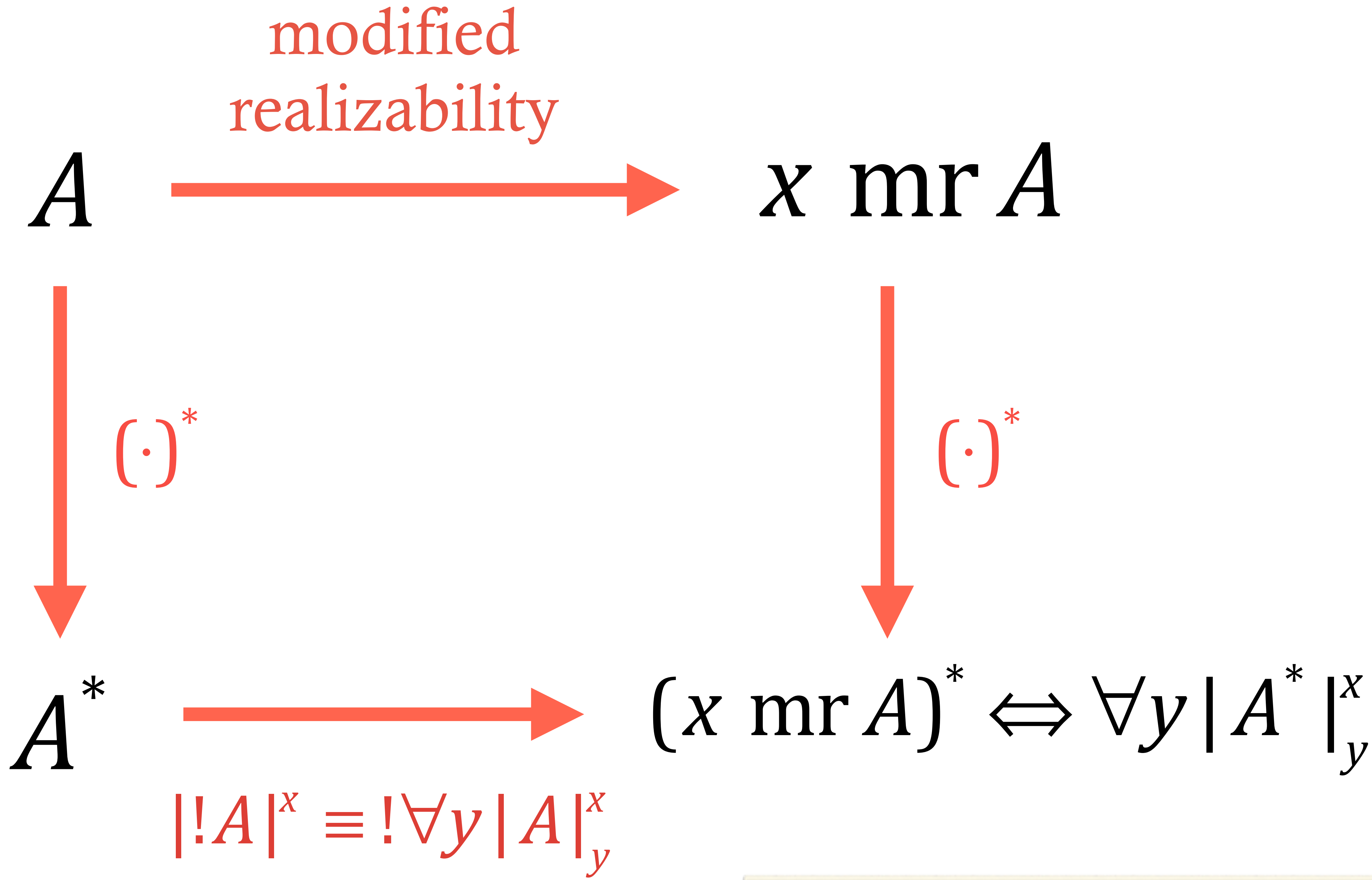
$$|\forall z\, A(z)|^f_{y,s} \equiv |A(s)|^{f(s)}_y$$

$$|\exists z\, A(z)|^{x,s}_y \equiv |A(s)|^x_y$$

based on earlier work of de Paiva and Shirahata

P. Oliva, **Modified realizability interpretation of classical linear logic**, LICS 2007

G. Ferreira and P. Oliva, **Functional interpretations of intuitionistic linear logic**, Logical Methods in Computer Science, 7(1), 2011



interpretations (only) differ in treatment of $!A$

!A	Trans.	Interpretation
$\|!A\|^x \equiv !\forall y \|A\|^x_y$	$(\cdot)^*$ or $(\cdot)^\circ$	Kreisel modified realizability
$\|!A\|^x_a \equiv !\forall y \in a \|A\|^x_y$	$(\cdot)^*$ or $(\cdot)^\circ$	Diller-Nahm interpretation
$\|!A\|^x_a \equiv !\|A\|^x_a$	$(\cdot)^*$ or $(\cdot)^\circ$	Gödel's Dialectica interpretation
$\|!A\|^x \equiv !\forall y \|A\|^x_y \otimes !A$	$(\cdot)^\circ$	modified realizability with truth
$\|!A\|^x \equiv !\forall y \|A\|^x_y \otimes !A$	$(\cdot)^*$	q-variant of modified realizability
$\|!A\|^x_a \equiv !\forall y \in a \|A\|^x_y \otimes !A$	$(\cdot)^\circ$	Diller-Nahm with truth

J. Gaspar and P. Oliva, **Proof interpretations with truth**, MLQ, 56(6):591-610, 2010

Part 3:

Applications

(*classical logic and games*)

How about classical logic, arithmetic and analysis?

law of excluded middle

$$A \vee \neg A$$

double negation elimination

$$\neg\neg A \rightarrow A$$

pre-linearity

$$(A \rightarrow B) \vee (B \rightarrow A)$$

Peirce's law

$$((A \rightarrow B) \rightarrow B) \rightarrow A$$

Markov principle

$$\neg \forall n D(n) \rightarrow \exists n \neg D(n)$$

Drinker's paradox

$$\exists x (D(x) \rightarrow \forall y D(y))$$

finite choice

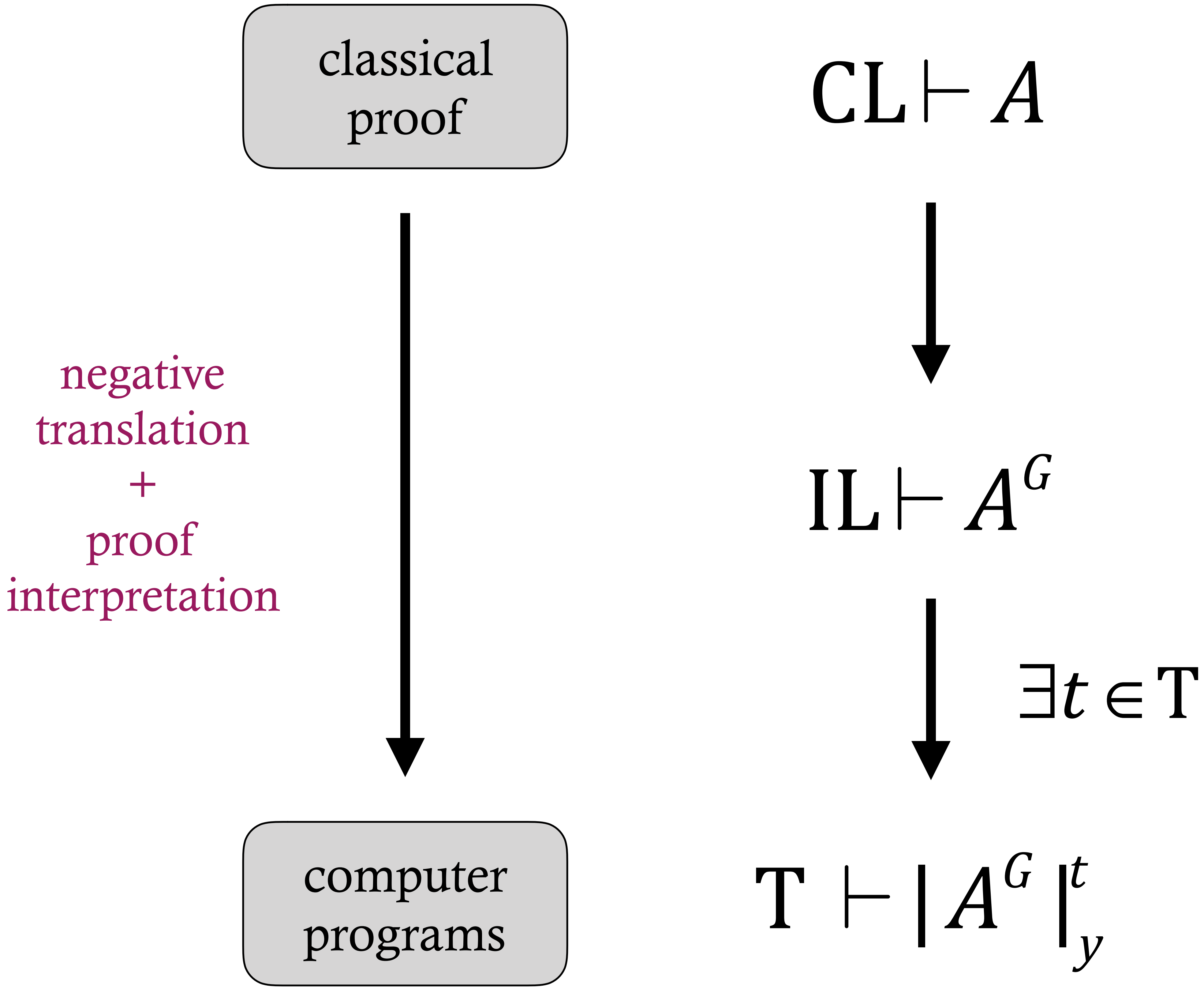
$$\forall n < k \exists i\, A(n,i) \rightarrow \exists s \forall n < k\, A(n, s_n)$$

Gödel-Gentzen translation		
$(P)^G$	$\equiv$	$\neg\neg P$
$(A \wedge B)^G$	$\equiv$	$A^G \wedge B^G$
$(A \vee B)^G$	$\equiv$	$\neg\neg(A^G \vee B^G)$
$(A \to B)^G$	$\equiv$	$A^G \to B^G$
$(\forall z\, A)^G$	$\equiv$	$\forall z\, A^G$
$(\exists z\, A)^G$	$\equiv$	$\neg\neg\exists z\, A^G$

Kuroda translation		
$(P)^K$	$\equiv$	P
$(A \wedge B)^K$	$\equiv$	$A^K \wedge B^K$
$(A \vee B)^K$	$\equiv$	$A^K \vee B^K$
$(A \to B)^K$	$\equiv$	$A^K \to B^K$
$(\forall z\, A)^K$	$\equiv$	$\forall z \neg\neg A^K$
$(\exists z\, A)^K$	$\equiv$	$\exists z\, A^K$

$$\mathrm{CL} \vdash A \quad \Rightarrow \quad \begin{array}{l} \mathrm{IL} \vdash A^G \\ \mathrm{IL} \vdash \neg\neg A^K \end{array}$$

G. Ferreira and P. Oliva, **On the relation between various negative translations**, Logic, Construction, Computation, vol 3, 227-258, 2012



classical proof
negative translation + proof interpretation
computer programs
$\mathrm{CL} \vdash A$
$\mathrm{IL} \vdash A^G$
$\exists t \in \mathrm{T}$
$\mathrm{T} \vdash |A^G|^t_y$

classical proof

$$\exists x^X \forall y^R A(x,y)$$

negative translation + proof interpretation

$$\neg\neg\exists x^X \forall y^R A(x,y)$$

program $\phi^{(X\to R)\to X}$?

computer programs

$$\forall p^{X\to R} A(\phi(p), p(\phi(p)))$$

player

program $\phi^{(X \to R) \to X}$

higher-order games explain higher-order programs

move outcome

$$\exists x^X \forall y^R A(x, y)$$

x is a good move given outcome *y*

game continuation

$$\forall p^{X \to R} A(\phi(p), p(\phi(p)))$$

optimal move optimal outcome

M. Escardó and P. Oliva, **Sequential games and optimal strategies**, Proc. of the Royal Society A, 467:1519-1545, 2011

Logical form	Specifies
$\exists x^X \forall y^R A(x,y)$	player
$\forall n \exists x^X \forall y^R A_n(x,y)$	sequence of players
finite choice (bounded collection)	finite game
countable choice	unbounded game

M. Escardó and P. Oliva, **Selection functions, bar recursion, and backward induction**, MSCS, 20 (2), pp .127-168, 2010

P. Oliva and T. Powell, **A game-theoretic computational interpretation of proofs in classical analysis**, Gentzen's Centenary, 501-531, 2015

Summary

- Realizability also has a **"relational" presentation**
- Relational presentation allows for **interpretation of LL** and **unification** (including truth variants)
- Classical proofs dealt with by combining interpretation with a **negative translation**
- Classical proof (and higher-order programs) can be "explained" in terms of **higher-order games**

- P. Oliva, **Unifying functional interpretations**, NDJFL, 47 (2), 2006
- P. Oliva, **Modified realizability interpretation of classical linear logic**, LICS 2007
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- G. Ferreira and P. Oliva, **On the relation between various negative translations**, Logic, Construction, Computation, vol 3, 227-258, 2012
- P. Oliva and T. Powell, **A game-theoretic computational interpretation of proofs in classical analysis**, Gentzen's Centenary, 501-531, Springer, 2015