Closure of System T under the Bar Recursion Rule

Paulo Oliva (joint work with S. Steila)

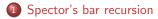
Queen Mary University of London

University Leeds Wednesday, 1 November 2017

・ロト ・ 一下・ ・ ヨト ・ ヨト

Э

Outline

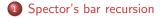








Outline



2 Schwichtenberg's proof





Spector's Bar Recursion

(1958) Gödel's Dialectica interpretation of arithmetic (system T)

イロン イボン イモン イモン 一日

4/21

- (1962) Spector extends interpretation to *analysis* (T + BR)
- (1968) Howard interpretation of bar induction (T + BR)
- (1971) Scarpellini shows C is a model of BR
- (1979) Schwichtenberg closure theorem (low types)
- (1981) Howard's ordinal analysis of BR (low types)
- (1985) Bezem shows \mathcal{M} is a model of BR

Spector's Bar Recursion (Rule)

Given $s: \tau^*$ let $\hat{s}: \tau^{\mathbb{N}}$ be the extension of s with 0's For each pair of types τ, σ , and given G, H and Y

$$\mathsf{BR}^{\tau,\sigma}(s) \stackrel{\sigma}{=} \begin{cases} G(s) & \text{if } Y(\hat{s}) < |s| \\ H(s)(\lambda x^{\tau}.\mathsf{BR}(s * x)) & \text{otherwise} \end{cases}$$

where

$$G : \tau^* \to \sigma$$

$$Y : \tau^{\mathbb{N}} \to \mathbb{N}$$

$$H : \tau^* \to (\tau \to \sigma) \to \sigma$$

・ロ・ ・雪・ ・ ヨ・

5/21

- Spector's bar recursion

Schwichtenberg's Closure Theorem

Theorem

System T is closed under the bar recursion rule when τ 's type level is either 0 or 1

That is, given G, H and Y terms in T, the functional

$$\mathsf{BR}^{\tau,\sigma}(s) \stackrel{\sigma}{=} \left\{ \begin{array}{ll} G(s) & \text{if } Y(\hat{s}) < |s| \\ H(s)(\lambda x^{\tau}.\mathsf{BR}(s * x)) & \text{otherwise} \end{array} \right.$$

イロン 不通 とうほう 不同 と

is also T definable

— Spector's bar recursion

Counter-example for $\tau > 1$

Howard (1968) showed that bar recursion of type ρ can be defined using the bar recursion rule of type $(\mathbb{N} \to \rho) \to \rho$



Spector's bar recursion

```
Counter-example for \tau > 1
```

Howard (1968) showed that bar recursion of type ρ can be defined using the bar recursion rule of type $(\mathbb{N} \to \rho) \to \rho$

Since bar recursion, even of type $\rho = \mathbb{N}$, is not T definable



Spector's bar recursion

Counter-example for $\tau > 1$

Howard (1968) showed that bar recursion of type ρ can be defined using the bar recursion rule of type $(\mathbb{N} \to \rho) \to \rho$

Since bar recursion, even of type $\rho = \mathbb{N}$, is not T definable

it follows that T is not closed under the bar recursion rule for $\tau=(\mathbb{N}\to\mathbb{N})\to\mathbb{N}$



Outline









Schwichtenberg's Proof

Published in The Journal of Symbolic Logic (1971)

"On bar recursion of type 0 and 1"

5 pages long (actual proof only two pages long)



1. Translate terms G, H, Y into infinitary terms (get rid of recursor)



- 1. Translate terms G, H, Y into infinitary terms (get rid of recursor)
- 2. Define a bar $S_Y(s) =$ "sequence s is secure for term Y"



- 1. Translate terms G, H, Y into infinitary terms (get rid of recursor)
- 2. Define a bar $S_Y(s) =$ "sequence s is secure for term Y"
- 3. Complement of $S_Y(s)$ is a tree

- 1. Translate terms G, H, Y into infinitary terms (get rid of recursor)
- 2. Define a bar $S_Y(s) =$ "sequence s is secure for term Y"
- 3. Complement of $S_Y(s)$ is a tree
- 4. See BR as a recursion on this tree

- 1. Translate terms G, H, Y into infinitary terms (get rid of recursor)
- 2. Define a bar $S_Y(s) =$ "sequence s is secure for term Y"
- 3. Complement of $S_Y(s)$ is a tree
- 4. See BR as a recursion on this tree
- 5. Define order-preserving embedding of tree into ε_0 -ordinals

イロト 不良 と 不良 と 一日

- 1. Translate terms G, H, Y into infinitary terms (get rid of recursor)
- 2. Define a bar $S_Y(s) =$ "sequence s is secure for term Y"
- 3. Complement of $S_Y(s)$ is a tree
- 4. See BR as a recursion on this tree
- 5. Define order-preserving embedding of tree into $\varepsilon_0\text{-}ordinals$

イロン イボン イモン イモン 一日

6. Hence, BR can be mimicked by ε_0 -ordinal recursion

- 1. Translate terms G, H, Y into infinitary terms (get rid of recursor)
- 2. Define a bar $S_Y(s) =$ "sequence s is secure for term Y"
- 3. Complement of $S_Y(s)$ is a tree
- 4. See BR as a recursion on this tree
- 5. Define order-preserving embedding of tree into $\varepsilon_0\text{-}\text{ordinals}$
- 6. Hence, BR can be mimicked by ε_0 -ordinal recursion
- 7. By Tait, we can find equivalent T definition of BR(s)

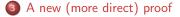


イロト 不良 と 不良 と 一日

Outline

Spector's bar recursion

2 Schwichtenberg's proof





Base case: $Y(\alpha)$ is constant

When $Y(\alpha)$ is constant n, BR becomes

$$\mathsf{BR}^{\tau,\sigma}(s) \stackrel{\sigma}{=} \left\{ \begin{array}{ll} G(s) & \text{if } |s| > n \\ H(s)(\lambda x^{\tau}.\mathsf{BR}(s * x)) & \text{if } |s| \le n \end{array} \right.$$



Base case: $Y(\alpha)$ is constant

When $Y(\alpha)$ is constant n, BR becomes

$$\mathsf{BR}^{\tau,\sigma}(s) \stackrel{\sigma}{=} \left\{ \begin{array}{ll} G(s) & \text{if } |s| > n \\ H(s)(\lambda x^{\tau}.\mathsf{BR}(s * x)) & \text{if } |s| \le n \end{array} \right.$$

It is easy to write down a T term (uniformly in G and H) computing the same function



Base case: $Y(\alpha)$ is constant

When $Y(\alpha)$ is constant n, BR becomes

$$\mathsf{BR}^{\tau,\sigma}(s) \stackrel{\sigma}{=} \left\{ \begin{array}{ll} G(s) & \text{if } |s| > n \\ H(s)(\lambda x^{\tau}.\mathsf{BR}(s * x)) & \text{if } |s| \le n \end{array} \right.$$

It is easy to write down a T term (uniformly in G and H) computing the same function

Needs primitive recursion of type $\tau^* \to \sigma$



Base case: $Y(\alpha)$ is constant

When $Y(\alpha)$ is constant n, BR becomes

$$\mathsf{BR}^{\tau,\sigma}(s) \stackrel{\sigma}{=} \left\{ \begin{array}{ll} G(s) & \text{if } |s| > n \\ H(s)(\lambda x^{\tau}.\mathsf{BR}(s * x)) & \text{if } |s| \le n \end{array} \right.$$

It is easy to write down a T term (uniformly in G and H) computing the same function

・ロ・ ・雪・ ・ ヨ・

Needs primitive recursion of type $\tau^* \to \sigma$

Let us refer to this T term as cBR

Proof Idea

Part 1: Show that BR is definable in "general BR" Part 2: Show that T is closed under "general BR" (first part works for any type, second part requires the type restriction)



General BR

For any $\ensuremath{\textit{bar}}\xspace S$ consider the defining equation

$$\mathsf{gBR}^{S}(s) \stackrel{\sigma}{=} \left\{ \begin{array}{ll} G(s) & \text{if } S(s) \\ H(s)(\lambda x^{\tau}.\mathsf{gBR}^{S}(s * x)) & \text{if } \neg S(s) \end{array} \right.$$



General BR

For any $\textit{bar}\ S$ consider the defining equation

$$\mathsf{gBR}^{S}(s) \stackrel{\sigma}{=} \begin{cases} G(s) & \text{if } S(s) \\ H(s)(\lambda x^{\tau}.\mathsf{gBR}^{S}(s * x)) & \text{if } \neg S(s) \end{cases}$$

Definition

We say that a bar S secures $Y \colon \tau^{\mathbb{N}} \to \mathbb{N}$ if for all s^{τ^*}

 $S(s) \Rightarrow \lambda \beta . Y(s * \beta)$ is constant



Part 1: BR definable in general BR

Theorem Fix $Y : \tau^{\mathbb{N}} \to \mathbb{N}$. The functional $\lambda G, H, s. BR^{\tau,\sigma}(G, H, Y)(s)$ is T-definable in gBR^S, for any bar S securing Y



Part 1: BR definable in general BR

Theorem Fix $Y : \tau^{\mathbb{N}} \to \mathbb{N}$. The functional $\lambda G, H, s. \mathsf{BR}^{\tau,\sigma}(G, H, Y)(s)$ is T-definable in gBR^S , for any bar S securing Y

Proof.

Use the bar S to spot when Y becomes constant, then apply the T construction for the case when Y is constant.



Part 2: Closure of T under gBR rule

Theorem

Fix a T-term $Y: \tau^{\mathbb{N}} \to \mathbb{N}$. For some S securing Y the functional gBR^S is T definable.



Part 2: Closure of T under gBR rule

Theorem

Fix a T-term $Y: \tau^{\mathbb{N}} \to \mathbb{N}$. For some S securing Y the functional gBR^S is T definable.

Proof.

(*Construction*) By induction on Y.

(*Correctness proof*) Use a logical relation to show that the constructed term is indeed equivalent to gBR^S .



The Construction (case $\tau = \mathbb{N}$)

Let $\mathbb{N}^{\circ} \equiv$ the type of gBR. We will map \mathbb{N} to \mathbb{N}° .

Let α be a special variable of type $\mathbb{N} \to \mathbb{N}$ (generic)

 $0^{\circ} = \lambda G.G$



The Construction (case
$$\tau = \mathbb{N}$$
)

Let $\mathbb{N}^{\circ} \equiv$ the type of gBR. We will map \mathbb{N} to \mathbb{N}° .

$$0^{\circ} = \lambda G.G$$

$$\mathsf{Succ}^\circ = \lambda \Phi^{\mathbb{N}^\circ} \cdot \Phi$$

The Construction (case $\tau = \mathbb{N}$)

Let $\mathbb{N}^\circ\equiv$ the type of gBR. We will map \mathbb{N} to $\mathbb{N}^\circ.$

$$0^{\circ} = \lambda G.G$$

$$\mathsf{Succ}^\circ = \lambda \Phi^{\mathbb{N}^\circ} \cdot \Phi$$

$$\alpha^{\circ} \qquad = \lambda \Phi^{\mathbb{N}^{\circ}} \lambda G. \Phi(\lambda s'. \mathsf{cBR}(G, Y(\hat{s'}))(s'))$$

The Construction (case $\tau = \mathbb{N}$)

Let $\mathbb{N}^\circ\equiv$ the type of gBR. We will map \mathbb{N} to $\mathbb{N}^\circ.$



The Construction (case
$$au = \mathbb{N}$$
)

Let $\mathbb{N}^\circ\equiv$ the type of gBR. We will map \mathbb{N} to $\mathbb{N}^\circ.$



The Construction (case $\tau = \mathbb{N}$)

Let $\mathbb{N}^\circ\equiv$ the type of gBR. We will map \mathbb{N} to $\mathbb{N}^\circ.$

Let α be a special variable of type $\mathbb{N} \to \mathbb{N}$ (generic)

The Construction (case
$$au = \mathbb{N}$$
)

Let $\mathbb{N}^\circ\equiv$ the type of gBR. We will map \mathbb{N} to $\mathbb{N}^\circ.$

Let α be a special variable of type $\mathbb{N} \to \mathbb{N}$ (generic)

(H can be fixed at outset, but extra work to remember Y)

The Construction: Recursor

Suppose $Y(\alpha) = \operatorname{Rec}(n_{\alpha}, x_{\alpha}, f_{\alpha})$



Suppose
$$Y(\alpha) = \operatorname{Rec}(n_{\alpha}, x_{\alpha}, f_{\alpha})$$

first ensure term n_{α} is secure (i.e. constant n)



Suppose $Y(\alpha) = \text{Rec}(n_{\alpha}, x_{\alpha}, f_{\alpha})$ first ensure term n_{α} is secure (i.e. constant n)

then ensure x_{α} is secure



Suppose $Y(\alpha) = \text{Rec}(n_{\alpha}, x_{\alpha}, f_{\alpha})$ first ensure term n_{α} is secure (i.e. constant n) then ensure x_{α} is secure and $f_{\alpha}(x_{\alpha})$ is secure



```
Suppose Y(\alpha) = \text{Rec}(n_{\alpha}, x_{\alpha}, f_{\alpha})
first ensure term n_{\alpha} is secure (i.e. constant n)
then ensure x_{\alpha} is secure
and f_{\alpha}(x_{\alpha}) is secure
```



```
Suppose Y(\alpha) = \text{Rec}(n_{\alpha}, x_{\alpha}, f_{\alpha})
first ensure term n_{\alpha} is secure (i.e. constant n)
then ensure x_{\alpha} is secure
and f_{\alpha}(x_{\alpha}) is secure
...
until f_{\alpha}^{n}(x_{\alpha}) is secure
```



```
Suppose Y(\alpha) = \text{Rec}(n_{\alpha}, x_{\alpha}, f_{\alpha})
first ensure term n_{\alpha} is secure (i.e. constant n)
then ensure x_{\alpha} is secure
and f_{\alpha}(x_{\alpha}) is secure
. . .
until f_{\alpha}^{n}(x_{\alpha}) is secure
can be done by induction hypothesis + primitive recursion
```



The Correctness Proof

Recall $\mathbb{N}^{\circ} \equiv$ the type of gBR Fix *H*. Define logical relation between T terms Base case:

$$f^{\mathbb{N}^{\circ}} \sim_{\mathbb{N}} g^{\mathbb{N}^{\mathbb{N}} \to \mathbb{N}} \equiv \exists S \text{ securing } g \text{ such that } f = \mathsf{g}\mathsf{BR}^S$$

The Correctness Proof

Recall $\mathbb{N}^{\circ} \equiv$ the type of gBR Fix *H*. Define logical relation between T terms Base case:

 $f^{\mathbb{N}^\circ}\sim_{\mathbb{N}} g^{\mathbb{N}^{\mathbb{N}}\to\mathbb{N}}\equiv \exists S \text{ securing }g \text{ such that }f=\mathrm{g}\mathrm{B}\mathrm{R}^S$

and, as usual:

$$f^{\rho_0^{\circ} \to \rho_1^{\circ}} \sim_{\rho_0 \to \rho_1} g^{\mathbb{N}^{\mathbb{N}} \to (\rho_0 \to \rho_1)}$$

$$\equiv \forall x^{\rho_0^{\circ}} \forall y^{\mathbb{N}^{\mathbb{N}} \to \rho_0} (x \sim_{\rho_0} y \to f(x) \sim_{\rho_1} \lambda \alpha. g(\alpha)(y\alpha))$$



Main Result

Theorem

Given a closed T term $Y \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, then $(Y\alpha)^{\circ} \sim Y$



Main Result

Theorem

Given a closed T term $Y \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$, then $(Y\alpha)^{\circ} \sim Y$

Proof.

By structural induction on Y



Main Result

Theorem

Given a closed T term
$$Y \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$$
, then $(Y\alpha)^{\circ} \sim Y$

Proof.

By structural induction on Y

Corollary

Fix $Y \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ in T. Then $\lambda G, H, s. BR(G, H, Y)(s)$ is T definable



Conclusion

Stronger result:

• Only Y needs to be T definable

More explicit construction:

 \bullet Given concrete Y, reasonably easy to find T definition of $\lambda G, H, s. {\rm BR}(G, H, Y)(s)$

Easy to calibrate T fragments:

• If Y is T_i then $\lambda G, H, s.BR(G, H, Y)(s)$ is in T_j , where $j = 1 + \max\{1, \text{level}(\sigma)\} + i$.

・ロト ・ 一下・ ・ ヨト ・ ヨト