Symmetric Bar Recursion

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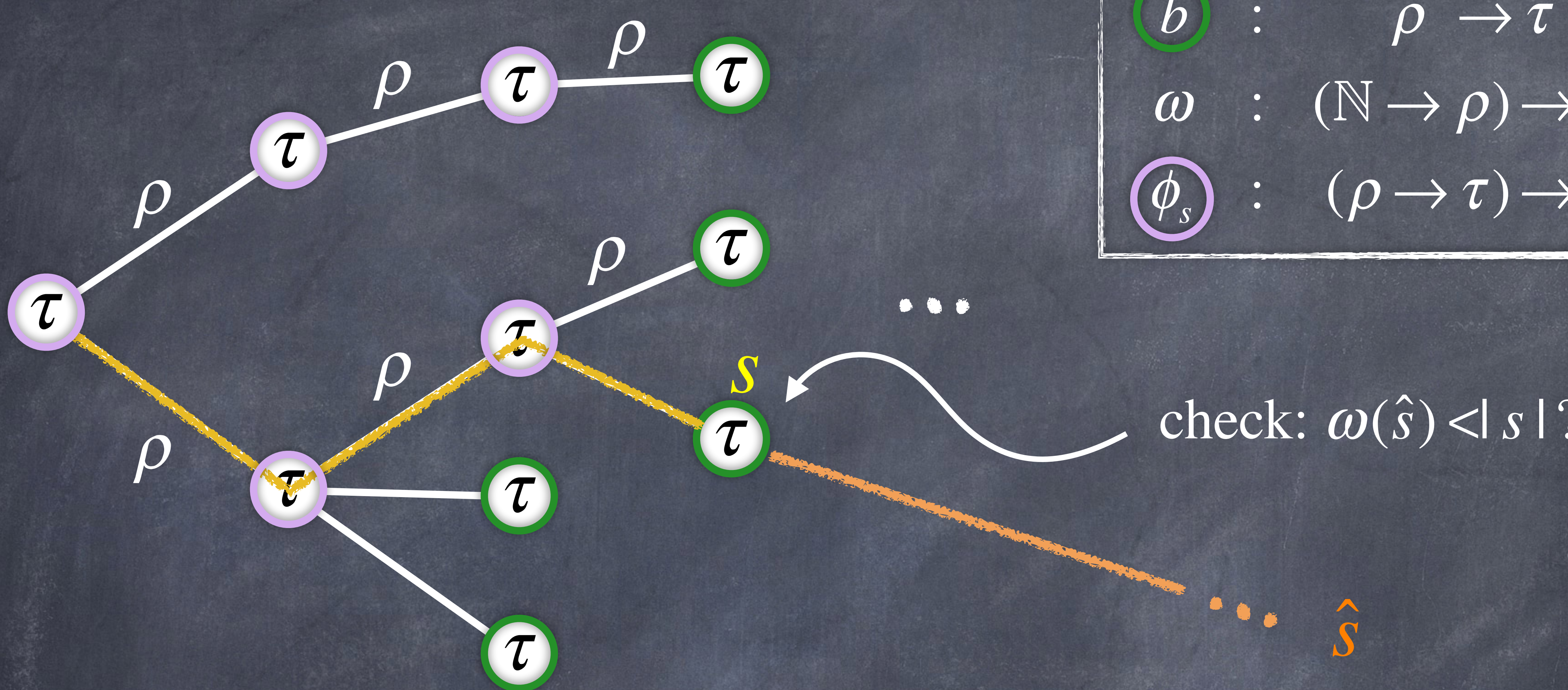
Spector's
Bar Recursion



b	:	$\rho^* \to \tau$
ω	:	$(\mathbb{N} \to \rho) \to \mathbb{N}$
ϕ_s	:	$(\rho \to \tau) \to \tau$

$$BR(s) \overset{\tau}{=} \begin{cases} b(s) & \text{if } \omega(\hat{s}) < |s| \\ \phi_s(\lambda x^{\rho}.BR(s * x)) & \text{otherwise} \end{cases}$$

Example of BR use:

Demo!

$$\forall H^{\mathbb{N}^{\mathbb{N}} \to \mathbb{N}} \exists \alpha, \beta^{\mathbb{N}^{\mathbb{N}}} \exists i^{\mathbb{N}} (\alpha i \neq \beta i \wedge H\alpha = H\beta)$$

Provable in classical analysis:

Classical Proof. As a simple case of the law of excluded middle (also known as the drinker's paradox) we have

$$\forall n^{\mathbb{N}} \exists \alpha^{\mathbb{N} \to \mathbb{N}} (\exists \beta (H\beta = n) \to H\alpha = n). \tag{5}$$

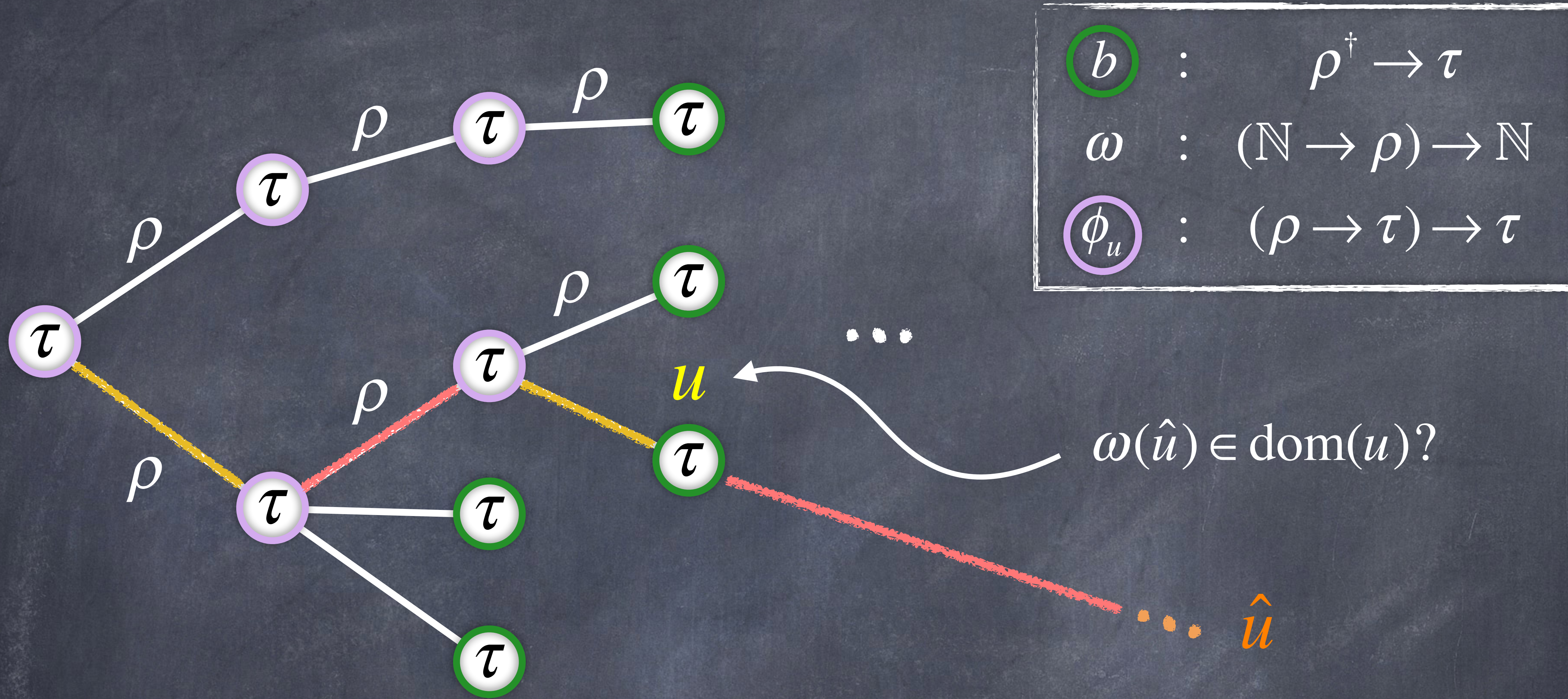
Applying $\mathsf{AC}_{\mathbb{N}, \mathbb{N} \to \mathbb{N}}$ to (5) yields a functional $f \colon \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ satisfying

$$\forall n (\exists \beta (H\beta = n) \to H(f(n)) = n). \tag{6}$$

The map f produces for each n a function $f(n) \colon \mathbb{N} \to \mathbb{N}$ such that whenever n is in the range of H, $f(n)$ maps to n. Now, define $\alpha_H := \lambda n. f(n)(n) + 1$ and let $i_H := H(\alpha_H)$ and $\beta_H := f(i_H)$. Then since i_H is in the range of H, by (6) we must have $H(\beta_H) = H(f(i_H)) = i_H = H(\alpha_H)$. But $\alpha_H \neq f(i_H) = \beta_H$. $\square$

Can compute witness using BR! [O'06]

Symmetric Bar Recursion



$$\mathrm{sBR}(u) \overset{\tau}{=} \begin{cases} b(u) & \text{if } \omega(\hat{u}) \in \mathrm{dom}(u) \\ \phi_u(\lambda x^{\rho} . \mathrm{sBR}(u \oplus (\omega(\hat{u}), x)) & \text{otherwise} \end{cases}$$

Definition: Given $u: \quad X^{\dagger}$

$\omega: \quad X^{\mathbb{N}} \to \mathbb{N}$

define $\{u\}_{\omega}: \ \mathbb{N} \to X^{\dagger}$ as

we call this the ω-thread of u

$$\{u\}_{\omega}(0) \triangleq \varnothing$$

$$\{u\}_{\omega}(i+1) \triangleq \begin{cases} \{u\}_{\omega}(i) \oplus (n, u(n)) & \text{if } n \in \mathrm{dom}(u) \\ \{u\}_{\omega}(i) & \text{otherwise} \end{cases}$$

where $n = \hat{\omega}(\{u\}_{\omega}(i))$

Definition: We say that u is an ω-thread if $u = \{u\}_{\omega}(\mathrm{dom}(u))$ (we write $S_{\omega}(u)$)

Symmetric Bar Induction

For all $\omega: X^{\mathbb{N}} \to \mathbb{N}$

(I) $\forall \alpha^{X^{\mathbb{N}}} \exists n^{\mathbb{N}} P(\{\alpha\}_{\omega}(n))$

(II) $\forall u \in S_{\omega} (\forall x^{X} P(u \oplus (n, x)) \to P(u))$

where $n = \omega(\hat{u})$

implies $P(\emptyset)$

Solving Spector's Equations

A symmetric solution to Spector's equations

Given ω, q and $\varepsilon_{(\cdot)}$ produce α, p and n such that

$$\begin{aligned} \omega(\alpha) &= n \\ \alpha(n) &= \varepsilon_n(p) \\ q(\alpha) &= p(\varepsilon_n(p)) \end{aligned}$$

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$$\Phi(u) = \begin{cases} u & \text{if } n_u \in \mathrm{dom}(u) \\ \Phi(u \oplus (n_u, a_u)) & \text{if } n_u \notin \mathrm{dom}(u) \end{cases}$$

where $n_u = \omega(\hat{u})$ and $p_u(x) = \hat{q}(\Phi(u \oplus (n_u, x))$ and $a_u = \varepsilon_n(p_u)$

$$u = \Phi(\emptyset) \qquad \alpha = \hat{u} \qquad n = \omega(\alpha) \qquad p = p_u$$

Inter-definability

- BR is T-definable from sBR (same type level)
- Change control function omega so that it always updates on the least undefined point
- This forces sBR to behave as BR

- sBR is T-definable from BR
- Quite a complicated construction
- Type level goes up by 2
- Main idea: Use values of function type to store state of computation

Conclusions

- A more symmetric variant of Spector's bar recursion
- Much more efficient in some cases
- T-equivalent to BR
- sBR more powerful than BR
  - sBR -> BR easy
  - BR -> sBR not so easy
- However, arguments by BI don't work, need version of BI based on finite partial functions