Modified Bar Recursion 15 years on…

(dedicated to Ulrich Berger)

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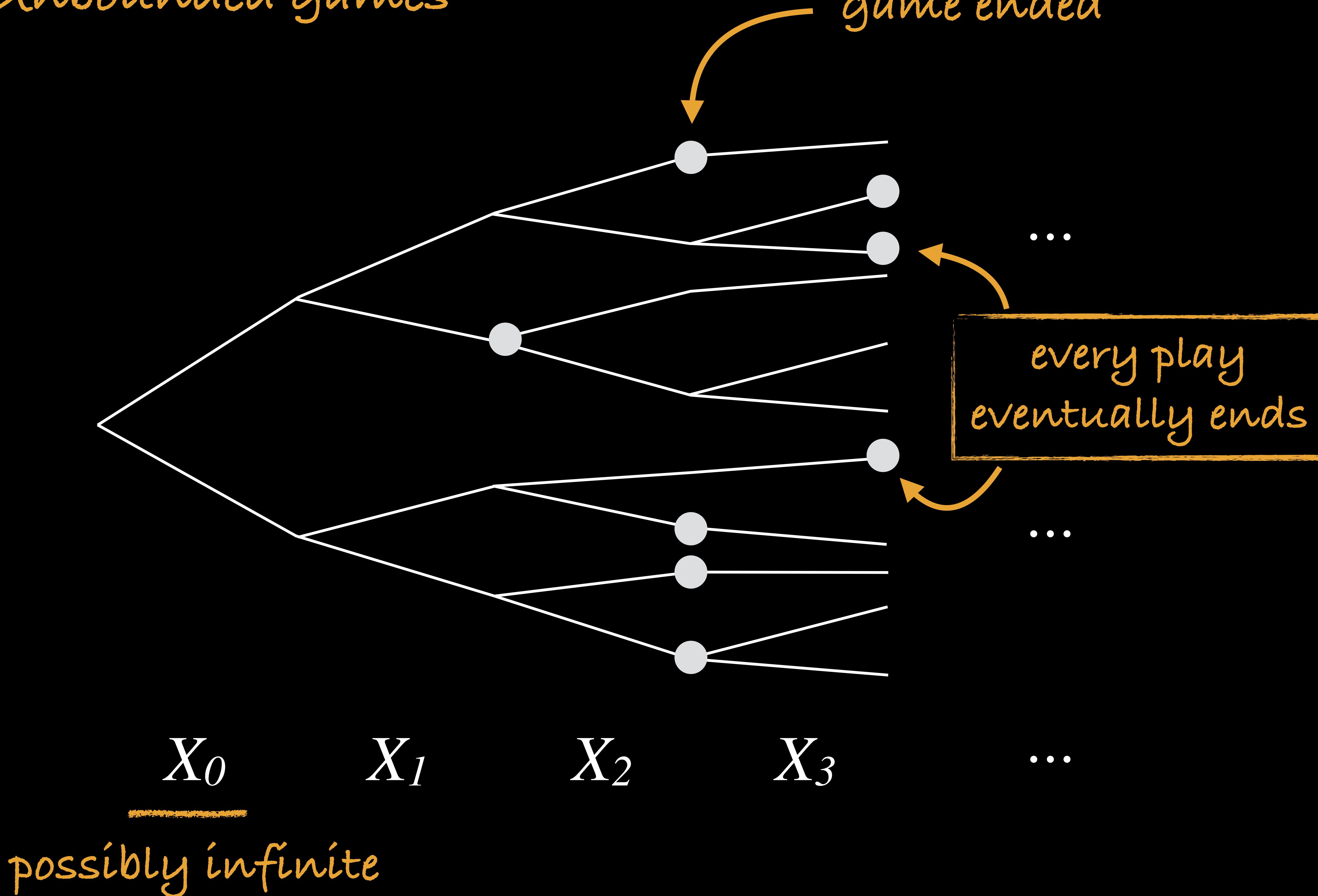
Paulo Oliva
Queen Mary University of London

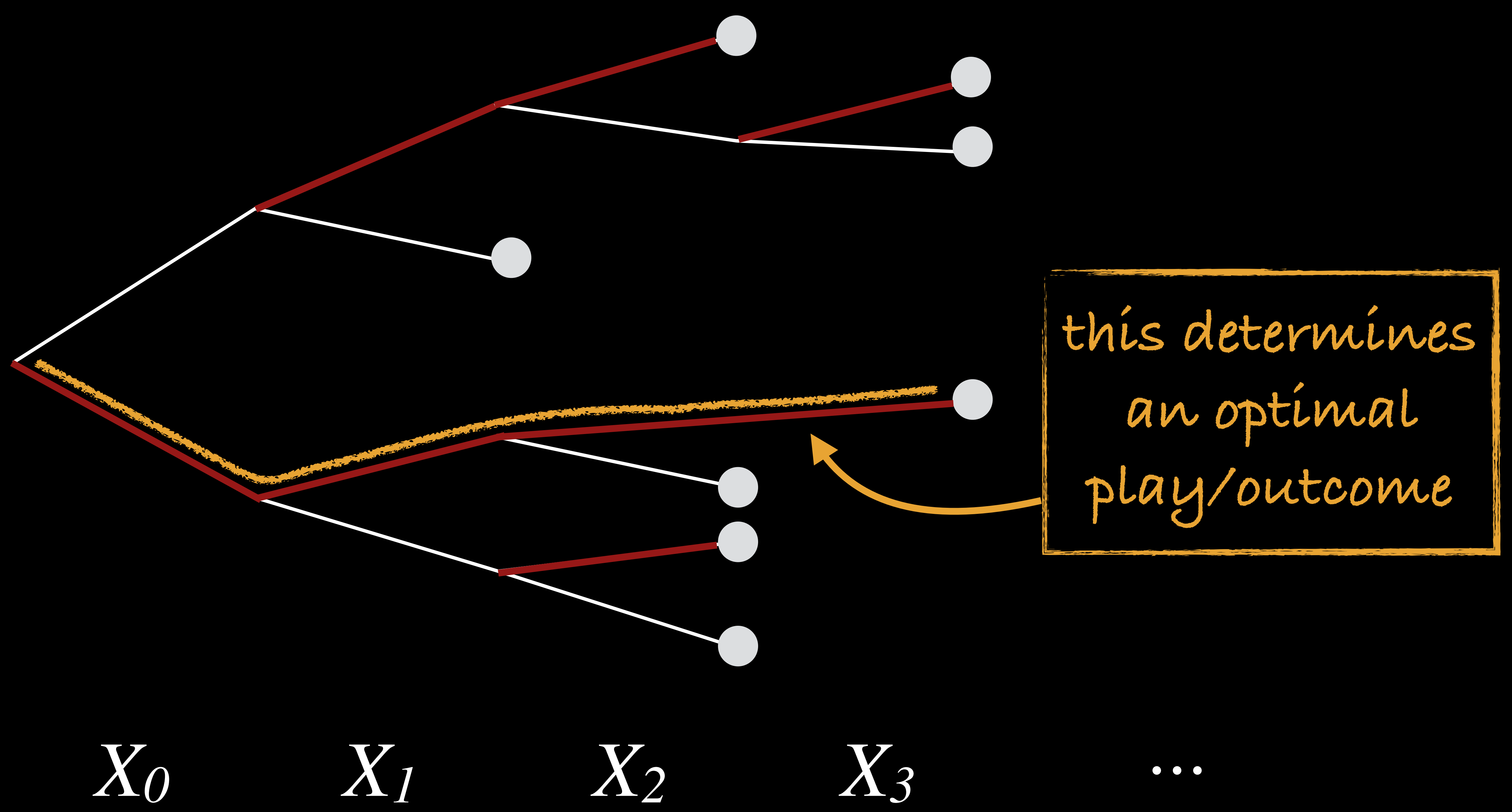
- Unbounded games
- Spector and modified bar recursion
- Selection functions
- Interpreting ineffective theorems via higher-order games
- The bar recursion zoo

Unbounded Games

Unbounded games



In such games we can define optimal strategies by starting from end nodes and working our way towards the start node



Optimal plays

$$\mathrm{OP} : \underset{\text{current play}}{\underline{X^*}} \to \underset{\text{optimal extension}}{\underline{X^*}}$$

$$\mathrm{OP}(s) = \begin{cases} [\,] & \text{if } s \text{ end of game} \\ m * \mathrm{OP}(s * m) & \text{otherwise} \end{cases}$$

$$m = \text{optimal\_move}(\underbrace{\lambda x.q(s * x * \mathrm{OP}(s * x))}_{\text{game continuation}})$$

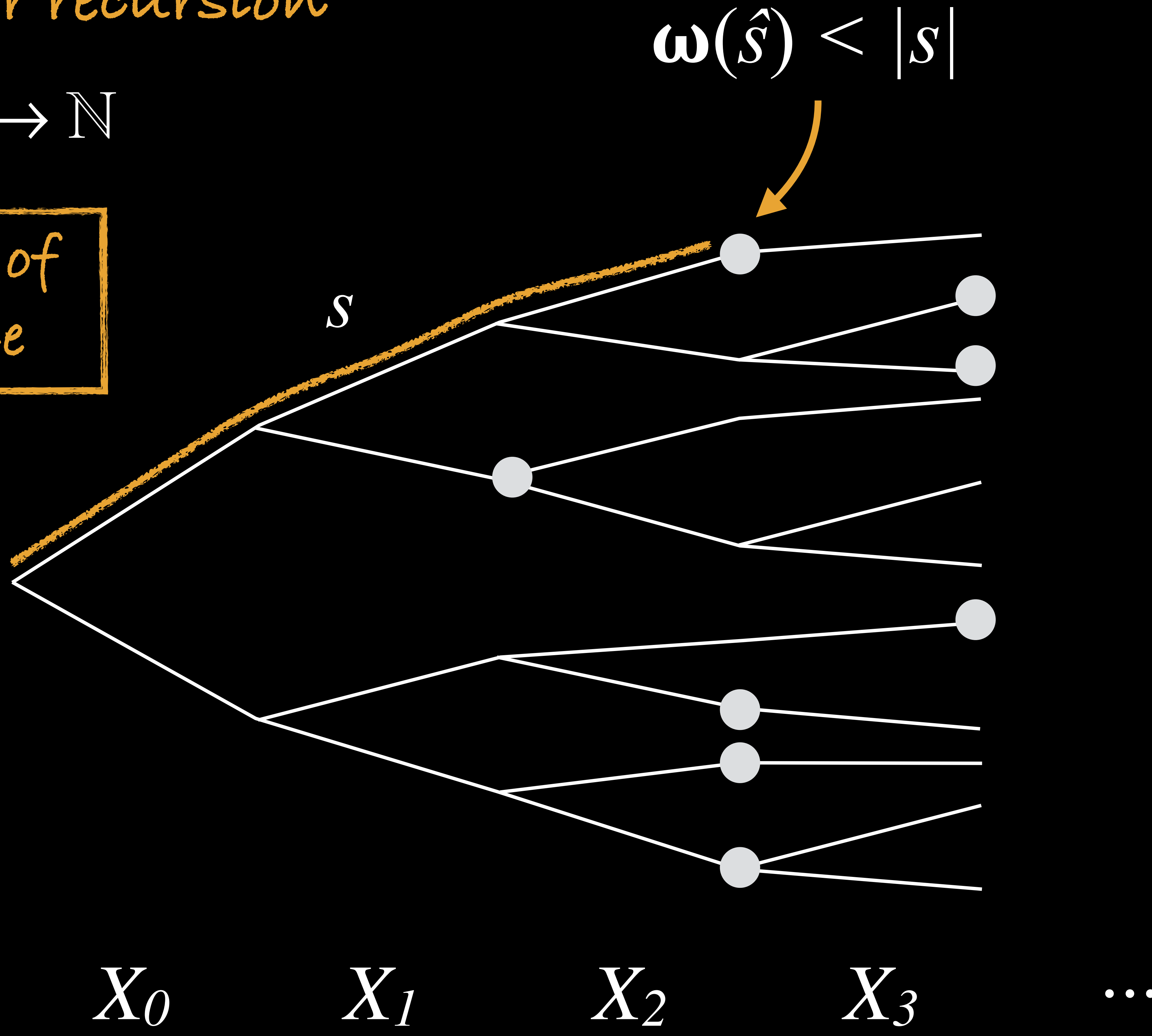
Bar Recursion

variants of bar recursion
mainly differ on the way
"end of game" is defined

Spector bar recursion

$$\boldsymbol{\omega} : \prod X_i \to \mathbb{N}$$

modulus of relevance



Spector bar recursion

$$\mathrm{SBR} : \underbrace{X^*}_{\text{current play}} \to \underbrace{X^*}_{\text{optimal extension}}$$

$$\mathrm{SBR}(s) = \begin{cases} [\,] & \text{if } \boldsymbol{\omega}(\hat{s}) < |s| \\ a * \mathrm{SBR}(s * a) & \text{otherwise} \end{cases}$$

$$a = \varepsilon_s(\underbrace{\lambda x. q(s * x * \mathrm{SBR}(s * x))}_{\text{game continuation}})$$

selection function

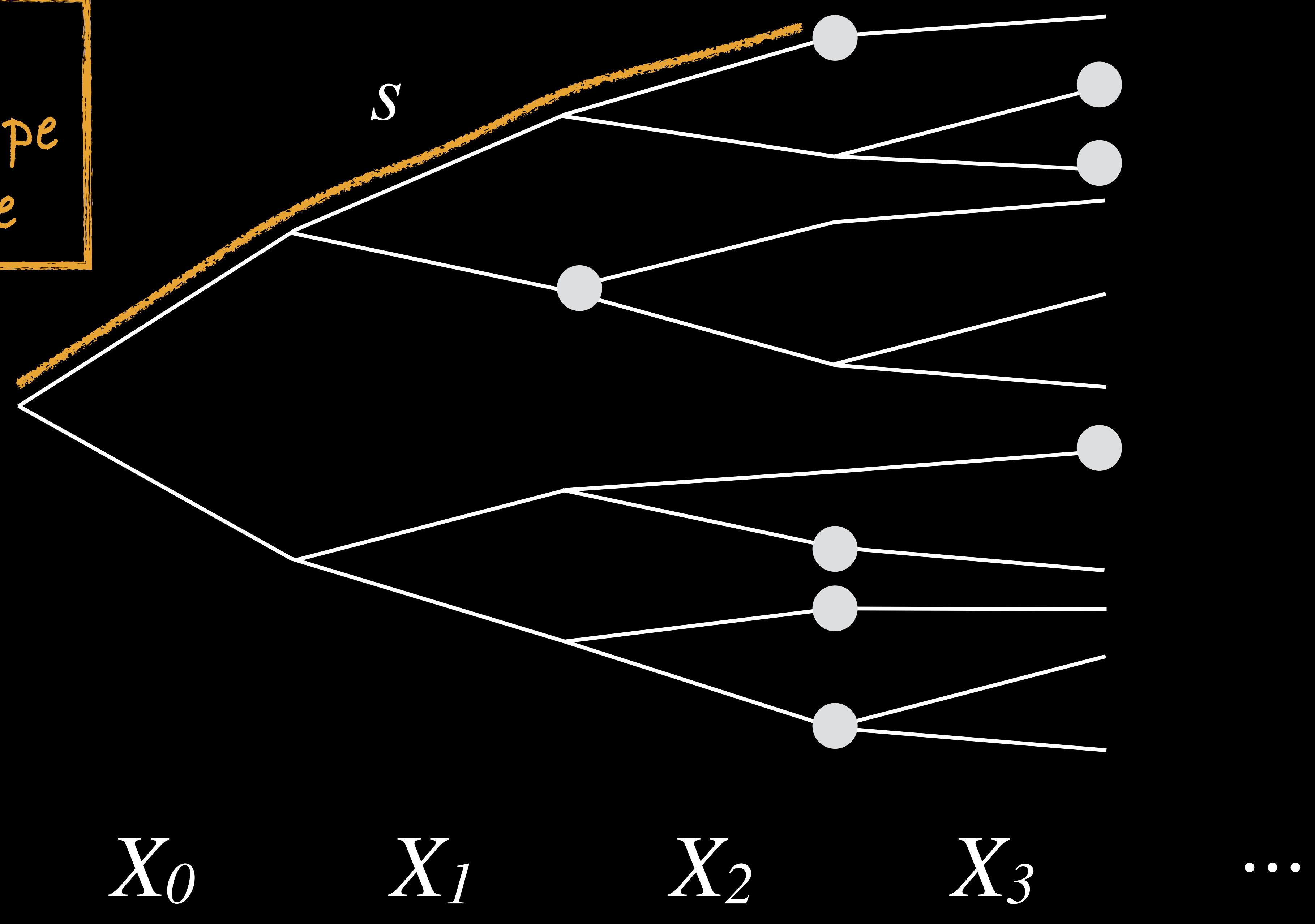
game continuation

Modified bar recursion

(implicit termination)

assumes outcome type is discrete

point where (continuous) outcome function can decide on outcome of game



Modified bar recursion

infinite stream

$$\mathrm{MBR} : X^* \to X^\omega$$

current play — optimal extension

$$\mathrm{MBR}(s) = a * \mathrm{MBR}(s * a)$$

continuous

$$a = \varepsilon_s(\lambda x. q(s * x * \mathrm{MBR}(s * x)))$$

selection function — game continuation

Selection Functions

It can be considerably easier to meet a request if you are told what it is needed for

Give me the heaviest object on earth! 😮

Give me the heaviest object on earth! 🤓
I need to keep this door shut.

$$\forall f \exists i \forall j \, (f(i) \leq f(j))$$

recursive? 😮

$$\forall f, p \exists i \, (f(i) \leq f(p(i)))$$

recursive? 🤓

yes!

Computational Interpretation of ineffective theorems via higher-order games

Consider the infinite pigeon-hole principle:

If you colour the natural numbers with finitely many colours then one colour must be used infinitely often

$$\forall n, c^{\mathbb{N}\to[n]}$$

$$\exists i < n$$

$$\forall j \exists k > j \,(c(k) = i)$$

colour i is used infinitely often

proof by induction and classical logic

recursive?

no!

$$\forall n, c^{\mathbb{N}\to[n]} \; \exists i < n \; \forall j \, \exists k > j \, (c(k) = i)$$

negative translation

$$\forall n, c^{\mathbb{N}\to[n]} \; \neg\forall i < n \, \neg \; \forall j \, \neg\forall k > j \, \neg(c(k) = i)$$

$$\forall n, c^{\mathbb{N}\to[n]}$$
$$\neg \forall i < n \neg$$
$$\forall j \neg \forall k > j \neg (c(k) = i)$$

$\downarrow$ dialectica interpretation

$$\forall n, c^{\mathbb{N}\to[n]} \, \varepsilon^{\mathbb{N}\to(\mathbb{N}\to\mathbb{N})\to\mathbb{N}}$$
$$\exists i < n \; \exists p^{\mathbb{N}\to\mathbb{N}}$$
$$p(\varepsilon_i \, p) > \varepsilon_i \, p \wedge c(p(\varepsilon_i \, p)) = i$$

selection functions = players

?

$$\forall n, c^{\mathbb{N}\to[n]}\, \varepsilon^{\mathbb{N}\to(\mathbb{N}\to\mathbb{N})\to\mathbb{N}}$$

$$\exists i < n\ \exists p^{\mathbb{N}\to\mathbb{N}}$$

game context

$$p(\varepsilon_i\, p) > \varepsilon_i\, p \wedge c(p(\varepsilon_i\, p)) = i$$

game outcome of given move

move of i-th player in context p

$$\forall n, c^{\mathbb{N}\to[n]}\ \varepsilon^{\mathbb{N}\to(\mathbb{N}\to\mathbb{N})\to\mathbb{N}}$$

$$\exists i < n\ \exists p^{\mathbb{N}\to\mathbb{N}}$$

$$p(\varepsilon_i\, p) > \varepsilon_i\, p \wedge c(p(\varepsilon_i\, p)) = i$$

Higher-order infinite pigeon-hole principle:

Given n players ε_i, $0<i<n$, and an assignment of players to numbers, $c : \mathbb{N}\to[n]$, there exists a player i and a game context p such that the outcome bounds the player's move, and the player is assigned to the outcome index

"In the field of analysis, it is common to make a distinction between "hard", "quantitative", or "finitary" analysis on one hand, and "soft", "qualitative", or "infinitary" analysis on the other.…The finitary version of an infinitary statement can be significantly more verbose and ugly-looking than the infinitary original, and the arrangement of quantifiers becomes crucial."

–Terence Tao
(https://terrytao.wordpress.com/2007/05/23/)

Bar Recursion Zoo

SBR (Spector'62)

KBR (Kohlenbach'89)

BBC (Berardi/Bezem/Coquand'99)

MBR (Berger'01)

IPS and EPS (Escardo/Oliva'10)

SBR =

KBR

BBC

MBR = =

IPS EPS

	majorizable	non-majorizable
S1-S9 computable	SBR = EPS	KBR
not S1-S9 computable	MBR = IPS = BBC	

Spector bar recursion (over finite partial functions)

$$\mathrm{sBR} : \underline{X^\dagger} \to \underline{X^\dagger}$$

finite partial function — optimal extension

$$\mathrm{sBR}(p) = \begin{cases} \emptyset & \text{if } n \in \mathrm{dom}(p) \\ u \oplus \mathrm{sBR}(s \oplus u) & \text{otherwise} \end{cases}$$

$$n = \boldsymbol{\omega}(p)$$

$$u = (n,\ \varepsilon_n(\lambda x. q(s \oplus (n,x) \oplus \mathrm{sBR}(s \oplus (n,x)))))$$

(O/Powell'15)

Herbrand bar recursion

$$\mathrm{HBR} : \underbrace{X^*}_{\text{finite sequence}} \to \underbrace{\mathcal{P}(X^*)}_{\text{finite set of extensions}}$$

$$\mathrm{HBR}(s) = \begin{cases} \{[\,]\} & \text{if } \omega(\hat{s}) < |s| \\ \{a * t \mid a \in A,\ t \in \mathrm{HBR}(s * a)\} & \text{else} \end{cases}$$

$$A = \varepsilon_s(\lambda x. q(s * x * \mathrm{HBR}(s * x)))$$

finite set of moves

(Escardo/O'15)

Open Questions

Relation between dialectica-stack and mr-stack?

Why does dialectica-stack require weaker BR?

Any principle for which mr-stack would require weaker recursion?

How about simultaneous games? (Hedges PhD)

Mixed strategies $\Longrightarrow$ Stochastic proof mining?

Is the rule version of MBR closed under system T?

Some references...

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