Logical Dependence via Functional Dependence

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Main point:

If we want to capture logical dependence via functional dependence in a modular fashion we must work with higher-types

logic

function

modular

higher-types

- From logic to functionals
- No-counterexample interpretation and the Dialectica interpretation
- Selection functions
- Higher-order game theory
- Pirates and coins

The axiom of choice expresses this translation from logical dependence to functional dependence

$$\forall x \exists y\, A(x,y) \rightarrow \exists f \forall x\, A(x, f\, x)$$

Truth/proof of premise implies the existence of a function witnessing the conclusion

$$\forall x \exists y\, A(x,y) \rightarrow \exists f \forall x\, A(x, f\, x)$$

- Is $A(x,y)$ decidable? Not necessarily
- Classical or constructive logic?

- What are the types of x and y? Any
- Existence of f in which universe/model?

Primitive recursive functionals

Consider the infinite pigeon-hole principle:

If you colour the natural numbers with finitely many colours then one colour must be used infinitely often

$$\forall\, n,\ c^{\mathbb{N}\to[n]}\ \exists\, i < n\ \forall\, j\ \exists\, k > j\ (c(k) = i)$$

recursive?

no!

proof by induction and classical logic

colour i is used infinitely often

$$\forall^{\mathbb{N}\to[n]}\, n, c$$

$$\exists\, i < n$$

$$\forall j \exists\, k > j\ (c(k) = i)$$

no counter-example interpretation

$$\forall^{\mathbb{N}\to[n]}\, n, c \quad \boxed{\forall^{\mathbb{N}\to\mathbb{N}}\, g}$$

counterexample function

$$\exists\, i < n \exists\, k > g(i)\ (c(k) = i)$$

recursive?

yes! (even primitive recursive)

no counter-example interpretation

- Kreisel (1951)

- Not modular! (Kohlenbach 1999)

Dialectica interpretation

- Gödel (1958) - ideas from 1930s

- Modular

- Interprets Markov principle, but not full CL

Dialectica interpretation + negative translation

$$\forall\, n, c^{\mathbb{N}\to[n]}$$

$$\exists\, i < n$$

$$\forall\, j \exists\, k > j\,(c(k) = i)$$

negative translation

$$\forall\, n, c^{\mathbb{N}\to[n]}$$

$$\neg \forall\, i < n \neg$$

$$\forall\, j \neg \forall\, k > j \neg (c(k) = i)$$

$$\forall\, n, c^{\mathbb{N}\to[n]}$$

$$\neg\forall\, i < n\,\neg$$

$$\forall\, j\neg\forall\, k > j\,\neg(c(k) = i)$$

dialectica intepretation

$$\forall\, n, c^{\mathbb{N}\to[n]}\ \varepsilon^{\mathbb{N}\to(\mathbb{N}\to\mathbb{N})\to\mathbb{N}}$$

$$\exists\, i < n\ \exists\, p^{\mathbb{N}\to\mathbb{N}}$$

$$p(\varepsilon_i\, p) > \varepsilon_i\, p \wedge c(p(\varepsilon_i\, p)) = i$$

selection functions = players

$$\forall\, n, c^{\mathbb{N}\to[n]}\ \varepsilon^{\mathbb{N}\to(\mathbb{N}\to\mathbb{N})\to\mathbb{N}}$$

$$\exists\, i < n\ \exists\, p^{\mathbb{N}\to\mathbb{N}}$$

?

game context

$$p(\varepsilon_i\, p) > \varepsilon_i\, p \wedge c(p(\varepsilon_i\, p)) = i$$

game outcome of given move

move of i-th player in context p

$$\forall\, n, c^{\,\mathbb{N}\to[n]}\ \varepsilon^{\,\mathbb{N}\to(\mathbb{N}\to\mathbb{N})\to\mathbb{N}}$$

$$\exists\, i < n\ \exists\, p^{\,\mathbb{N}\to\mathbb{N}}$$

$$p(\varepsilon_i\, p) > \varepsilon_i\, p \wedge c(p(\varepsilon_i\, p)) = i$$

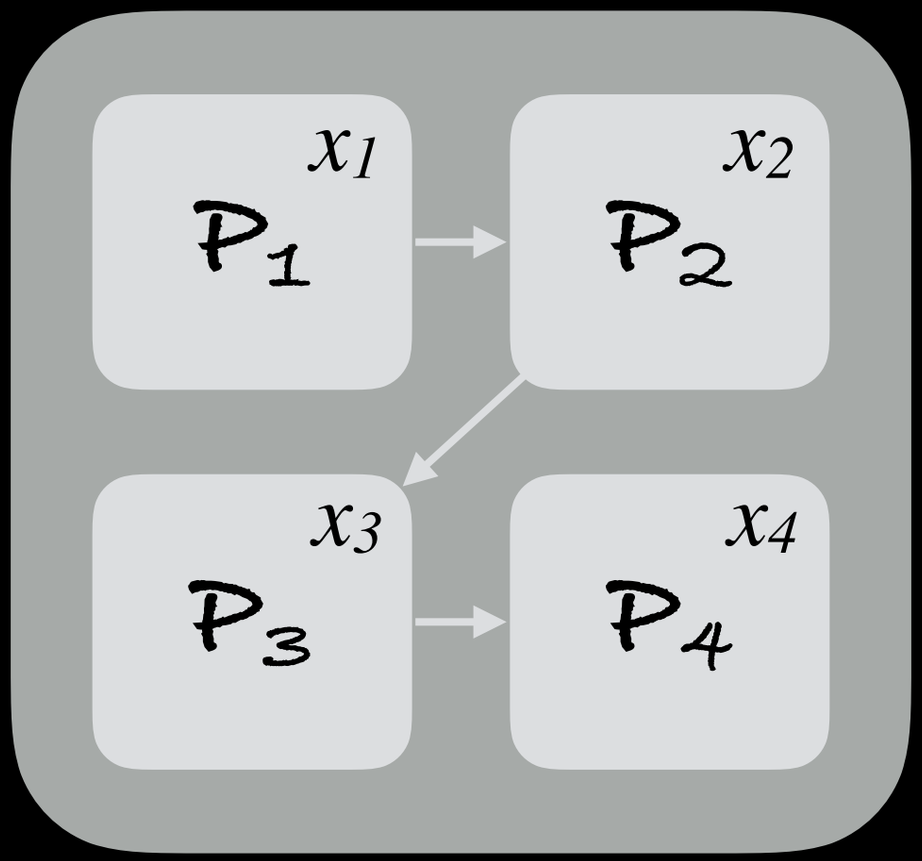
Higher-order infinite pigeon-hole principle:

Given n players ε_i, $0<i<n$, and an assignment of players to numbers, $c : \mathbb{N}\to[n]$, there exists a player i and a game context p such that the outcome bounds the player's move, and the player is assigned to the outcome index

"In the field of analysis, it is common to make a distinction between "hard", "quantitative", or "finitary" analysis on one hand, and "soft", "qualitative", or "infinitary" analysis on the other....The finitary version of an infinitary statement can be significantly more verbose and ugly-looking than the infinitary original, and the arrangement of quantifiers becomes crucial."

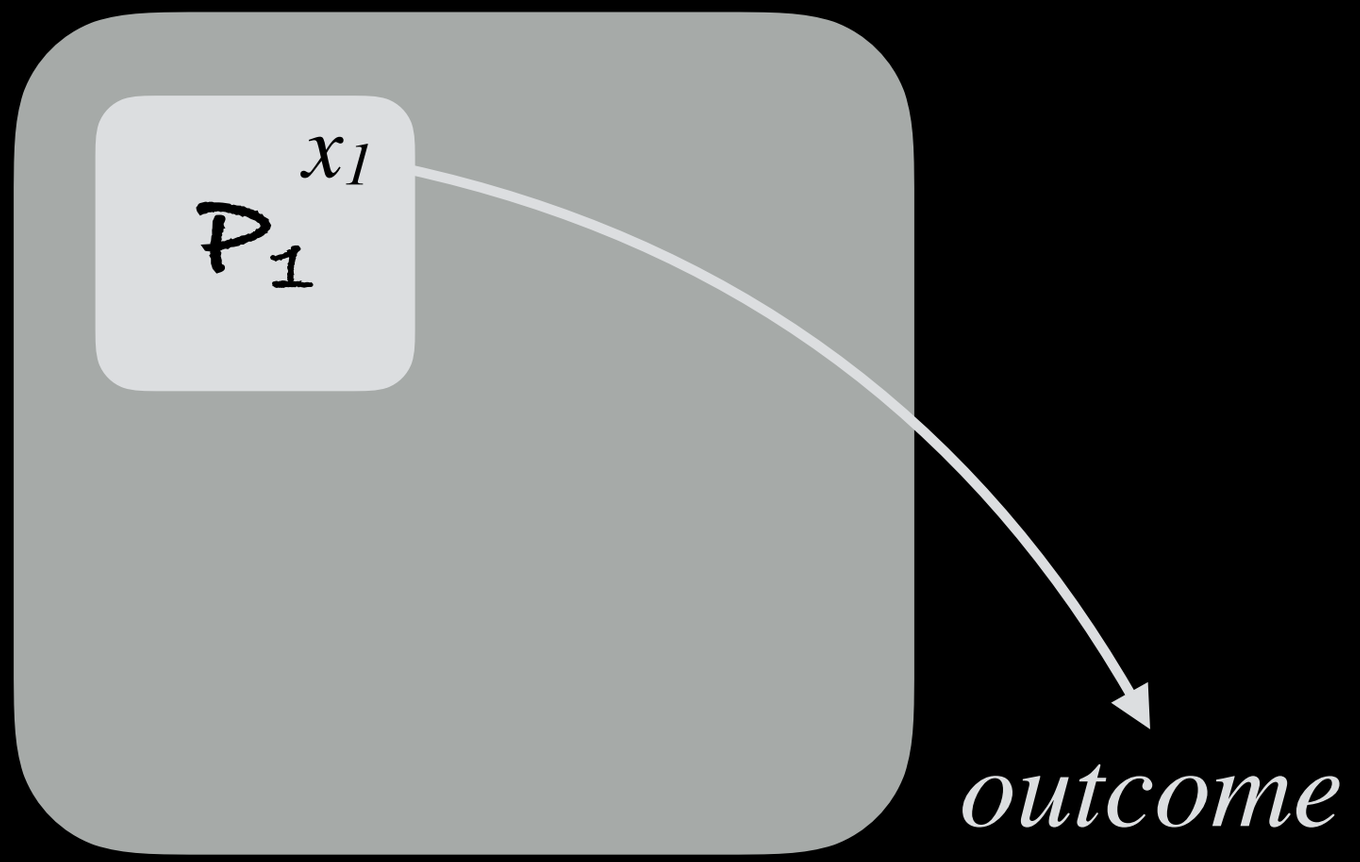
–Terence Tao
(https://terrytao.wordpress.com/2007/05/23/)

Higher-order Games



$$outcome = q(x_1, x_2, x_3, x_4)$$

Higher-order Games



context = mapping from move to outcome

player = mapping from context to best moves

context = mapping from move to outcome

player = mapping from context to best moves

$$\text{move} = X$$

$$\text{outcome} = R$$

$$\text{context} = X \to R$$

$$\text{player} = \underbrace{(X \to R) \to X}_{J_R X}$$

selection functions

selection functions $J_R X$

return :: $X \to J_R X$

a monad!

bind :: $J_R X \to (X \to J_R Y) \to J_R Y$

sequence :: $[J_R X] \to J_R [X]$

Surprising fact: sequence is total even on infinite lists! (for discrete types R)

Pirate Example

Pirates and Treasures[1]

A group of 7 pirates has 100 gold coins

They have to decide amongst themselves how to divide the treasure, but must abide by pirate rules:

- The most senior pirate proposes the division
- All of the pirates (including the most senior) vote on the division
  - If half or more vote for the division, it stands
  - If less than half vote for it, they throw the most senior pirate overboard and start again
- The pirates are perfectly logical, and entirely ruthless (only caring about maximising their own share of the gold)

What division should the most senior pirate suggest to the other six?

[1] http://www.ox.ac.uk/admissions/undergraduate/applying-to-oxford/interviews/sample-interview-questions

modelling the problem

outcome of the game? partition of n coins amongst pirates

moves of players?

when most senior: partition of n coins amongst k remaining pirates

when less senior: agree with proposed partition or not?

types...

```haskell
-- Share contains the distribution of coins for each pirate
type Pirate = Int

-- Share contains the distribution of coins for each pirate
type Share = [Int]

-- Whether a pirate agrees with share
type Vote = Bool

-- Collection of all votes
type Poll = [Vote]
```

```haskell
-- Number of coins
nc = 5

-- Number of pirates
np = 5
```

helpful functions...

```haskell
-- All possible ways to divide n coins amongst i pirates
divide :: Int -> Int -> [Share]
divide n 1 = [[n]]
divide n i = [ k:xs | k <- [0..n], xs <- divide (n-k) (i-1) ]
```

```haskell
-- Generic argmax
argmax :: (Ord b) => [a] -> (a -> b) -> [a]
argmax d p = [ x | x <- d, p x == maximum [ p x | x <- d ] ]
```

```haskell
-- Generic argmax with parallel map
argmax :: (NFData a, NFData b, Ord a, Ord b) => [a] -> (a -> b) -> [a]
argmax d p = [ x | (v,x) <- graph, v == (fst.last $ graph) ]
   where graph = sort $ parMap rdeepseq (\x -> (p x, x)) d
```

Basic player 1: The voter

Input

- Pirate index i
- Continuation p :: Bool → Share

Choose boolean that maximises his share

```haskell
v :: Pirate -> (Bool -> Share) -> Bool
v i p = head $ argmax [True,False] ((!!i).p)

sv :: Pirate -> J Share Bool
sv i = J (v i)
```

Basic player 2: The sharer

Input

- Pirate index i
- Continuation $p :: Share \to Share$

Choose global share that maximises his share

```haskell
s :: Pirate -> (Share -> Share) -> Share
s i p = head $ argmax dom ((!!i).p)
 where shares = divide nc (np - i)
       dom = map ((replicate i 0)++) shares

ss :: Int -> J Share Share
ss i = J (s i)
```

Composing players...

Poll player = sequencing of voters

```
sp :: Pirate -> J Share Poll
sp i = sequence (map sv [(i+1)..(np-1)])
```

Round player = Product of share and poll players

```
e :: Int -> J Share (Share, Poll)
e i = prod (ss i, sp i)
```

Global player = Sequence of round players

```
g :: J Share [(Share, Poll)]
g = sequence (map e [0..(np-1)])
```

Outcome function

Determines outcome of the game

Given list of [(Share, Poll)] return first share that was agreed in poll

If none, take last share

```haskell
q :: [(Share, Poll)] -> Share
q [(s,v)] = s
q ((s,v):xs) = if 1 + pro >= con then s else q xs
  where [con,pro] = (map length).group.sort $ True:False:v
```

Demo

- Number of pirates = 5
- Number of coins = 5

```
MBP{paulo}: ghc -O2 -threaded --make pirates.hs
[1 of 1] Compiling Main             ( pirates.hs, pirates.o )
Linking pirates ...

MBP{paulo}: time ./pirates +RTS -N
[3,0,1,0,1]

real 19m5.150s
user 138m18.802s
sys  4m19.996s
```

Optimal division proposed by most senior pirate

Modularity...

Solution built of individual/independent parts

More pirates / coins
(change values of np / nc)

Changes in voting rule
e.g. 1/3 enough
(change outcome function q)

Changes in "goal" of pirates
e.g. happy with 1 coin
(change selection functions)

Some references...

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