Substructural Logics

(*algebras and Weihrauch reducibility*)

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Motivation

Weihrauch Reducibility is “resource sensitive”

Linear Logic is a “resource sensitive” logic

Weihrauch Reducibility ≈ Linear Logic

Obs. Algebraic semantics associate to each logic **L** a class of algebras **A** such that **A** $\vDash x = \top$ iff **L** $\vdash [x]$

Project

1. Discover the “algebra” of the Weihrauch lattice

2. Design corresponding “proof system”

3. Show that $B \leq_W A$ iff $A \rightarrow B = \top$ iff $A \vdash B$

Substructural Logics

"Real" logic minus (some) structural rules

weakening (W)

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}$$

contraction (C)

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}$$

exchange (E)

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

Substructural Logics

Lambek logic = ban all three

Linear logic = ban weakening and contraction

Affine logic = ban contraction

Relevance logic = ban weakening

#connectives $\propto$ (*#str rules*)$^{-1}$

No C ⇒ 2 Conjunctions

conjunction 1

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$

(multiplicative)

conjunction 2

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}$$

(additive)

Resource “Interpretation”

$\Gamma \vdash A \,\&\, B$

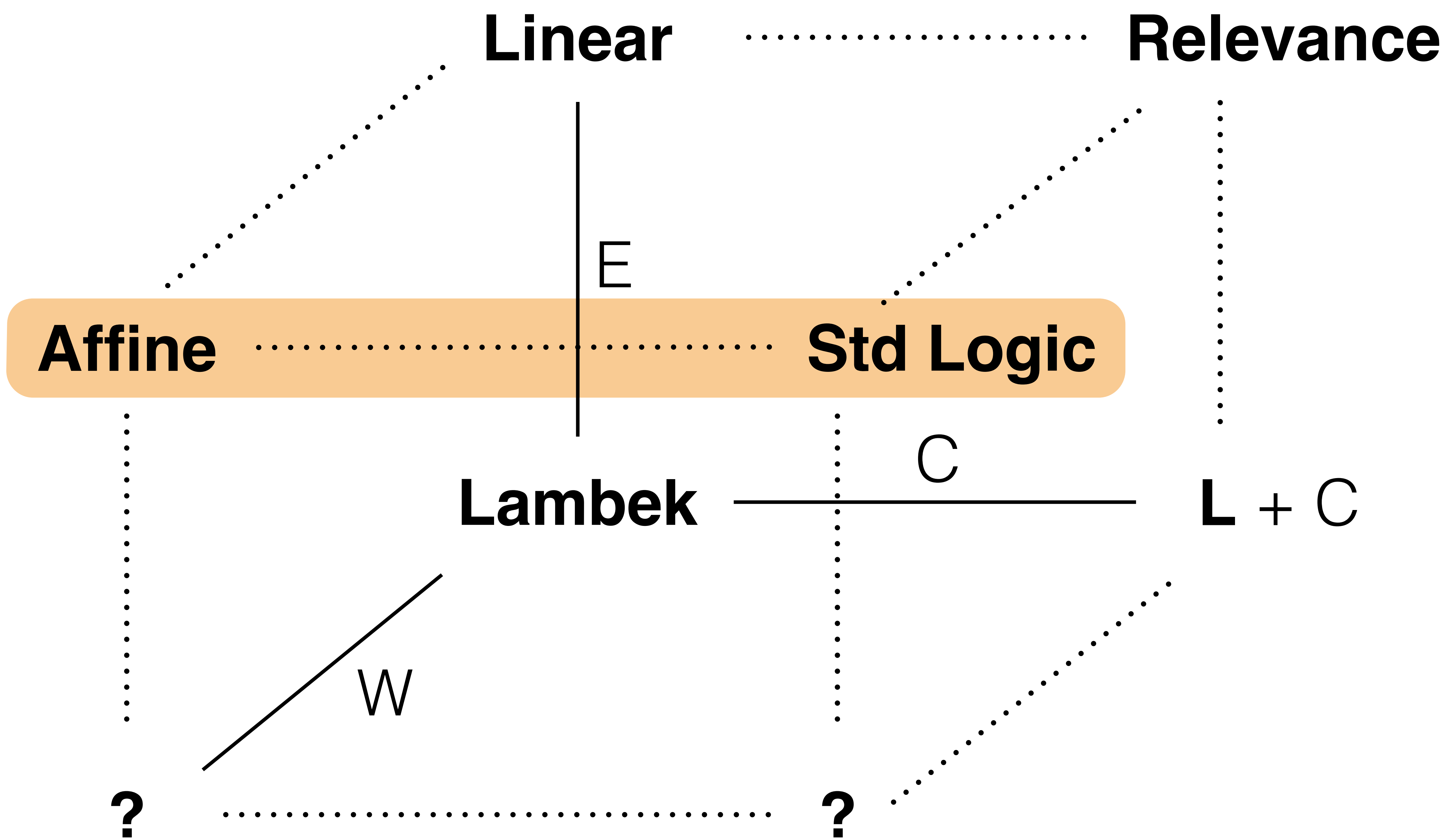
With Γ one can obtain **any** of A and B

$\Gamma \vdash A \otimes B$

With Γ one can obtain **both** A and B

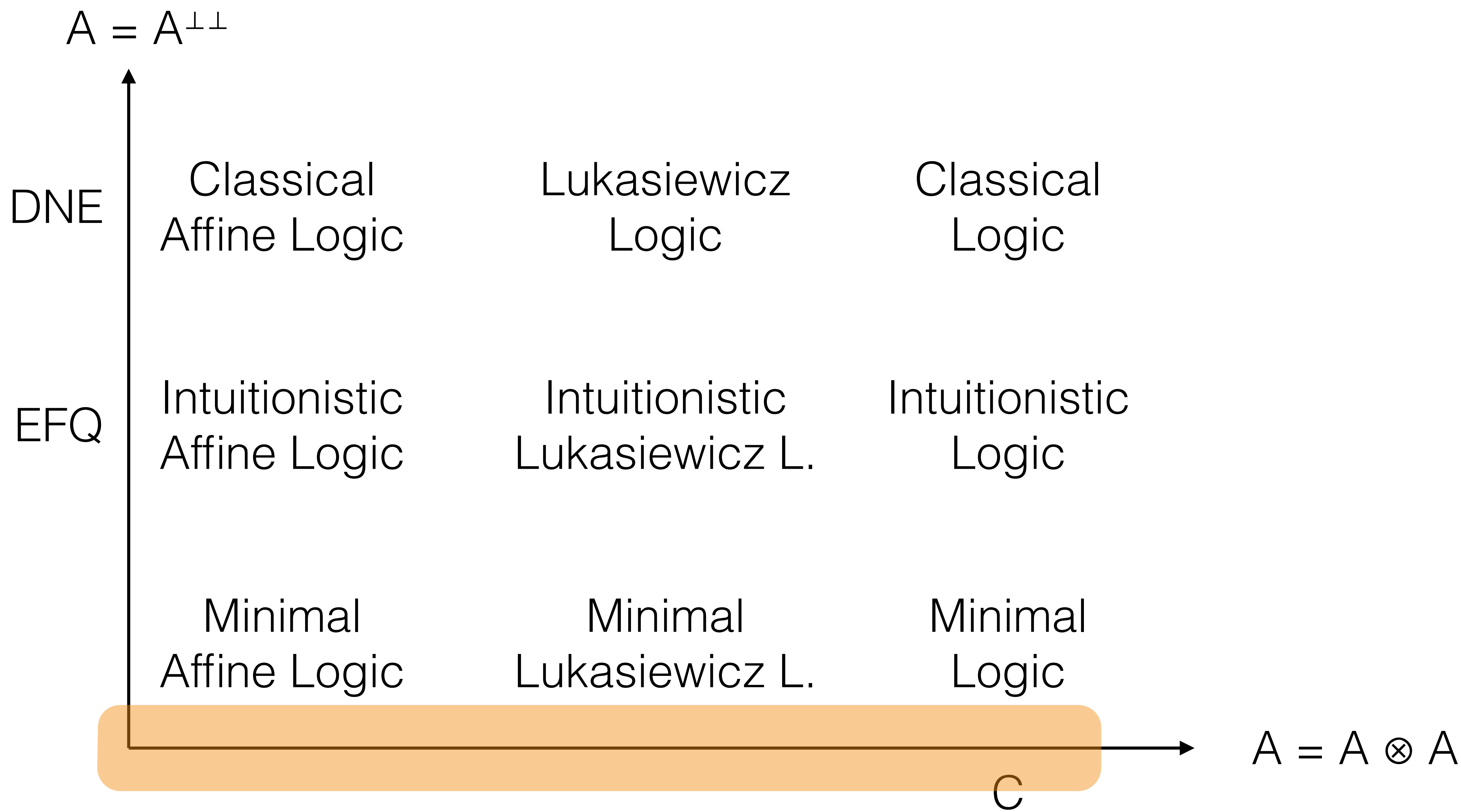
Logic of Degrees?

Weihrauch	Logic
$\sqcap$	&
$\sqcup$	$\oplus$
$\times$	$\otimes$
?	$\multimap$
$\star$	$\cdot$ (Lambek)
$\rightarrow$	$\backslash$ (Lambek)
A^*	$!_{DN}\, A$
$\hat{A}$	$!_{Kr}\, A$



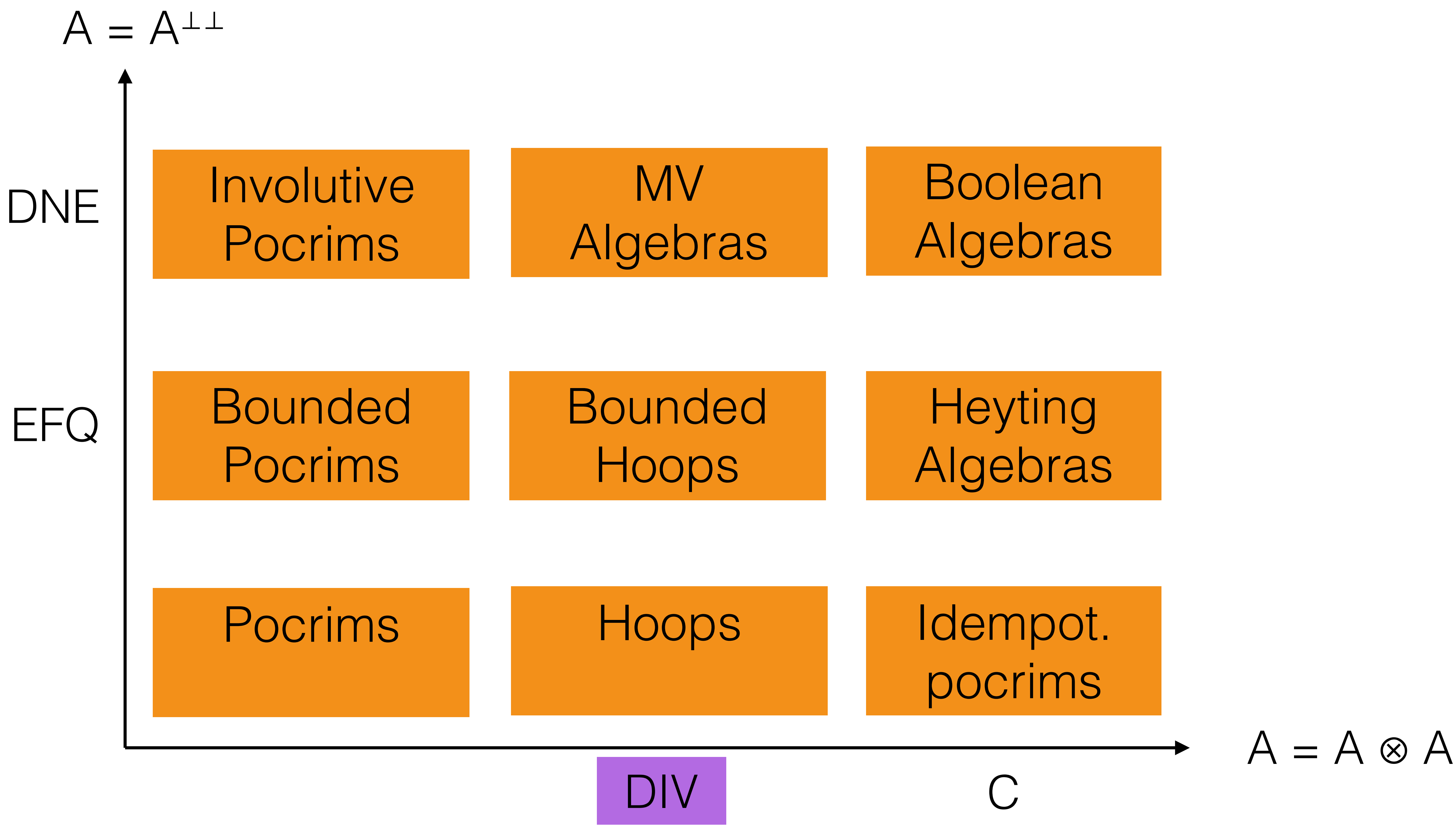
intuitionistic or classical?

C x DNE



$A = A^{\perp\perp}$
DNE
Classical Affine Logic
Lukasiewicz Logic
Classical Logic
EFQ
Intuitionistic Affine Logic
Intuitionistic Lukasiewicz L.
Intuitionistic Logic
Minimal Affine Logic
Minimal Lukasiewicz L.
Minimal Logic
$A = A \otimes A$
C

Algebras



A = A⊥⊥
DNE
EFQ
Involutive Pocrims
MV Algebras
Boolean Algebras
Bounded Pocrims
Bounded Hoops
Heyting Algebras
Pocrims
Hoops
Idempot. pocrims
A = A ⊗ A
DIV
C

Divisibility Axiom

Theorem [Arthan, O.'2014]. Tfae over pocrims / AL_m

(a) If $A \geq B$ then $A = B \otimes (B \to A)$

(b) $A \otimes (A \to B) = B \otimes (B \to A)$

(c) $A = ((A \to B) \to B) \otimes (((A \to B) \to B) \to A)$

Theorem [Arthan, O.'2014]. All usual negative translations "work" for Lukasiewicz logic

References

- Arthan and Oliva. *Negative translations for affine and Lukasiewicz logic*. Under review, 2015
- Arthan and Oliva. *On affine logic and Lukasiewicz logic* http://arxiv.org/abs/1404.0570
- Arthan and Oliva. *On pocrims and hoops* http://arxiv.org/abs/1404.0816
- Arthan and Oliva. *(Dual) hoops have unique halving* McCune Festschrift, LNAI 7788, pp. 165-180, 2013