On Proof Interpretations and Linear Logic

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Heyting Day

Utrecht, 27 February 2015

To Anne Troelstra and Dick de Jongh

Overview

$$\mathsf{IL} \xrightarrow[\text{interpretations}]{\cdots} \mathsf{IL}^\omega$$

IL = Intuitionistic predicate logic

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$$\mathsf{ILL}_p \longrightarrow \mathsf{ILL}_p^{\omega}$$

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ILL_p = Intuit. predicate linear logic (!-free)

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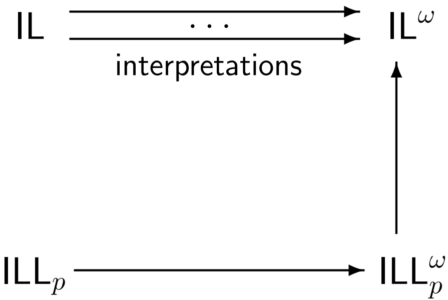

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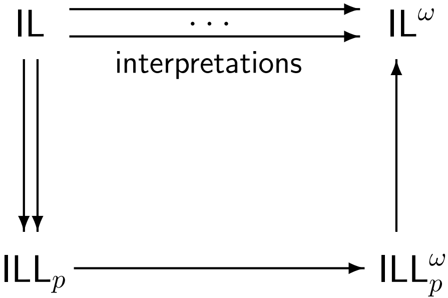

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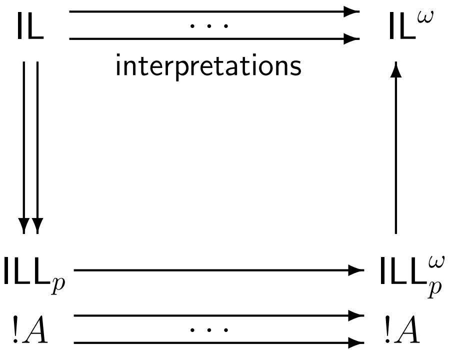

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Outline

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1. Proof Interpretations
2. Linear Logic
3. Unified Interpretation of Linear Logic

Formulas as Sets of Proofs

- **Formulas** as **set of proofs**

$$A \quad \mapsto \quad \{\pi \, : \, \pi \vdash A\}$$

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- Brouwer-Heyting-Kolmogorov Interpretation
- **Formulas** as **specifications** (**set of programs**)

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Formulas as Types

- Curry-Howard Correspondence
- Minimal Logic + Simply-typed λ-calculus

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- Extension 1: **Heyting arith. + Gödel primitive rec.**
- Extension 2: **Classical logic + Continuation operator**

$$\dfrac{\begin{array}{c}\neg A \\ \vdots \\ \bot\end{array}}{A} \qquad\qquad \dfrac{\begin{array}{c}A \\ \vdots \\ B\end{array} \qquad \begin{array}{c}\neg A \\ \vdots \\ B\end{array}}{B}$$

Functional Interpretations

Formula A $\Rightarrow$ **Set of functionals** $|A|$

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Proof $\pi \Rightarrow$ **Functional** $f_\pi \in |A|$

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Proof π $\Rightarrow$ **Functional** $f_\pi \in |A|$

$S_1 \vdash_\pi A$ $\Rightarrow$ $S_2 \vdash f_\pi \in |A|$

Functional Interpretations

Formula A	$\Rightarrow$	**Set of functionals** $\|A\|$
$\forall x \exists y (y > x)$		$\{f \; : \; fx > x\}$
Proof π	$\Rightarrow$	**Functional** $f_\pi \in \|A\|$

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Functional Interpretations

Formula A	$\Rightarrow$	**Set of functionals** $\lvert A \rvert$
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Proof π	$\Rightarrow$	**Functional** $f_\pi \in \lvert A \rvert$
$\ldots$		$\lambda x . x + 1$
$S_1 \vdash_\pi A$	$\Rightarrow$	$S_2 \vdash f_\pi \in \lvert A \rvert$

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Dialectica (Gödel 1958)

- *Relative consistency* of Peano arithmetic PA

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$$\begin{cases} \text{Falsity interpreted as empty set } (|\perp| \equiv \emptyset) \\ \mathsf{PA} \vdash \perp \quad \Rightarrow \quad \mathsf{T} \vdash \exists f (f \in \emptyset) \end{cases}$$

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Modified realizability (Kreisel 1959)

- *Independence results* for Heyting arithmetic HA

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Modified realizability (Kreisel 1959)

- *Independence results* for Heyting arithmetic HA

$$\begin{cases} |P| \text{ set of non-computable functionals} \\ \mathsf{HA} \vdash P \quad \Rightarrow \quad \mathsf{HA}^\omega \vdash \exists f (f \in |P|) \end{cases}$$

Functional Interpretations

- *Relative consistence*

  $S \vdash \bot \quad \Rightarrow \quad T \vdash \bot$

- *Independence results*

  $S \not\vdash P$

Functional Interpretations

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  $S \vdash \perp \quad \Rightarrow \quad T \vdash \perp$

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  $S + P \vdash A \quad \Rightarrow \quad S \vdash A$

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  $S + P \vdash A \quad \Rightarrow \quad S \vdash A$
- *Proof mining*
  $S + A \vdash B \quad \Rightarrow \quad S + A_{\mathsf{w}} \vdash B_{\mathsf{s}}$

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- *Proof mining*

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- *Build models*

  $\mathcal{M} \models S$

Other Interpretations

Diller-Nahm (1974)

- Decidability of prime formulas not required

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q-realizability (Kleene 1969 / Troelstra 1971)

- $t \,\mathsf{qr}\, (A \to B) \equiv \forall x((x \,\mathsf{qr}\, A) \underline{\wedge A} \to tx \,\mathsf{qr}\, B)$
- Related to Kleene's slash translation (1962)

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Realizability with truth (Smorynski / Troelstra 1998)

- $t \,\mathsf{rt}\, (A \to B) \equiv \forall x(x \,\mathsf{rt}\, A \to tx \,\mathsf{rt}\, B) \underline{\wedge (A \to B)}$
- Related to Aczel's slash translation (1968)

Troelstra on Proof Interpretations

(1971) A. S. Troelstra. **Notions of realizability for intuitionistic analysis**
2nd Scandinavian Logic Symposium, pp. 369 – 405

(1973) A. S. Troelstra. **Metamathematical Investigation of Intuitionistic Arithmetic and Analysis**
Lecture Notes in Mathematics 344

(1988) A. S. Troelstra and D. van Dalen. **Constructivism in Mathematics**
North-Holland, Amsterdam. Two volumes

(1998) A. S. Troelstra. **Realizability**
Handbook of Proof Theory, pages 408–473

Outline

Linear Logic (Girard 1987)

- Explicit treatment of **contraction**

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \quad \Rightarrow \quad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$$

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$$A \to B \quad \equiv \quad !A \multimap B$$

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- **Refinement** of logical connectives

	conjunction	disjunction
additive	$\sqcap$	$\sqcup$
multiplicative	$\star$	$+$

Intuitionistic Linear Logic

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \sqcap B} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \sqcup B}$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \star B} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall z A} \qquad \frac{\Gamma \vdash A[t/z]}{\Gamma \vdash \exists z A}$$

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Structural Rules

$$A \vdash A \quad \text{(id)}$$

$$\frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B} \text{(cut)}$$

$$\frac{\Gamma \vdash A}{\pi\{\Gamma\} \vdash A} \text{(per)}$$

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Exponential $!A$

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ (\text{con}) \qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ (\text{wkn})$$

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!\text{I}) \qquad \frac{\Gamma \vdash !A}{\Gamma \vdash A}\ (!\text{E})$$

Girard Translations

- Translation $(\cdot)^* : \mathsf{IL} \to \mathsf{ILL}$

$$
\begin{array}{rclcrcl}
(A \wedge B)^* & \equiv & A^* \sqcap B^* & & (\forall x A)^* & \equiv & \forall x A^* \\
(A \vee B)^* & \equiv & !A^* \sqcup !B^* & & (\exists x A)^* & \equiv & \exists x !A^* \\
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Thm. $\mathsf{IL} \vdash A$ iff $\mathsf{ILL} \vdash A^*$ iff $\mathsf{ILL} \vdash A^\circ$

Outline

One-Move Two-Player Games

- Game $G \equiv (D_1, D_2, R \subseteq D_1 \times D_2)$
- **Two players**
  Eloise and Abelard
- **Two domains of moves**
  $x \in D_1$ and $y \in D_2$
- **Adjudication of Winner**
  Relation $R(x, y)$ between players' moves
  (*usually* $|G|^x_y$)

Examples

Domain 1	Domain 2	Adjudication
$x \in \{0, 1, 2\}$	$y \in \{0, 1, 2\}$	$x + 1 = y \text{ mod } 3$

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$x \in \{0, \ldots, 5\}$	$y \in \{0, \ldots, 5\}$	$x + y$ is even

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$x \in \{0,1,2\}$	$y \in \{0,1,2\}$	$x+1=y$ mod 3
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$x \in \mathbb{N}$	$y \in \mathbb{N}$	$x \geq y$

Examples

Domain 1	Domain 2	Adjudication
$x \in \{0, 1, 2\}$	$y \in \{0, 1, 2\}$	$x + 1 = y$ mod 3
$x \in \{0, \ldots, 5\}$	$y \in \{0, \ldots, 5\}$	$x + y$ is even
$x \in \mathbb{N}$	$y \in \mathbb{N}$	$x \geq y$
$f \in \mathbb{N} \to \mathbb{N}$	$y \in \mathbb{N}$	$f(y) \geq y$

Goal

A is true (is provable)

iff

Eloise has winning move in game $|A|_y^x$

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Eloise has winning move in game $|A|_y^x$

$$A \quad \text{iff} \quad \exists x \forall y |A|_y^x$$

Interpretation

Formula A mapped to game $|A|_y^x$

Interpretation

Formula A mapped to game $|A|^x_y$

Inductively: Assume $|A|^x_y$ and $|B|^v_w$ defined, then

$$|A \star B|^{x,v}_{y,w} \quad :\equiv \quad |A|^x_y \star |B|^v_w$$

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$$|A \star B|^{x,v}_{y,w} \quad :\equiv \quad |A|^x_y \star |B|^v_w$$

$$|A \multimap B|^{f,g}_{x,w} \quad :\equiv \quad |A|^x_{fxw} \multimap |B|^{gx}_w$$

Interpretation

Formula A mapped to game $|A|_y^x$

Inductively: Assume $|A|_y^x$ and $|B|_w^v$ defined, then

$$
\begin{aligned}
|A \star B|_{y,w}^{x,v} &:\equiv |A|_y^x \star |B|_w^v \\
|A \multimap B|_{x,w}^{f,g} &:\equiv |A|_{fxw}^x \multimap |B|_w^{gx} \\
|\forall z A(z)|_{y,a}^{f} &:\equiv |A(a)|_y^{fa}
\end{aligned}
$$

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|\forall z A(z)|^{f}_{y,a} \quad &:\equiv \quad |A(a)|^{fa}_y \\
|\exists z A(z)|^{x,a}_{y} \quad &:\equiv \quad |A(a)|^{x}_y
\end{aligned}
$$

Soundness

Theorem

If

$$\mathsf{ILL} \overset{\pi}{\vdash} A$$

there is t *(extracted from* π*) such that*

$$\mathsf{ILL}^{\omega} \vdash \forall y |A|^t_y$$

Exponential Game $!A$

Opponent can choose a “set” a of moves

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(1) Singleton

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(2) Finite

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(2) Finite

$|!A|^x_a \quad :\equiv \forall y \in a \, |A|^x_y$

(3) Arbitrary

$|!A|^x \quad :\equiv \forall y |A|^x_y$

Interpretation

Given $A \mapsto |A|^x_y$

Interpretation

Given $A \mapsto |A|^x_y$, possible interpretations of $!A$

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$$|!A|^x \;:\equiv\; !\forall y |A|^x_y \star !A$$

Interpretation

Given $A \mapsto |A|^x_y$, possible interpretations of $!A$

$$|!A|^x_y \quad :\equiv \quad !|A|^x_y \qquad \text{(Dialectica)}$$

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(Dialectica)

(Diller-Nahm)

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$$|!A|^x \quad :\equiv \quad !\forall y |A|^x_y \qquad \text{(realizability)}$$

$$|!A|^x \quad :\equiv \quad !\forall y |A|^x_y \star !A$$

Interpretation

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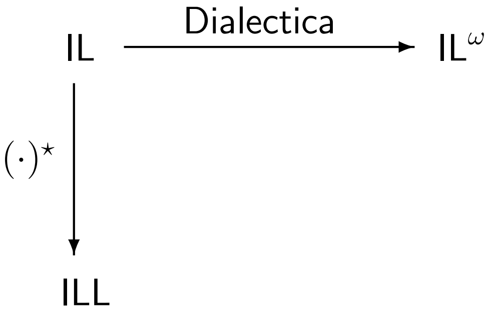
$$|!A|^x \;:\equiv\; !\forall y |A|^x_y \qquad \text{(realizability)}$$

$$|!A|^x \;:\equiv\; !\forall y |A|^x_y \star !A \qquad \text{(q- or truth-real.)}$$

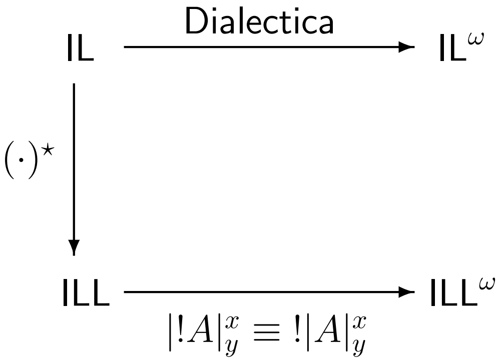
Interpretations of IL

$$\mathrm{IL} \xrightarrow{\text{Dialectica}} \mathrm{IL}^{\omega}$$

Interpretations of IL



Interpretations of IL



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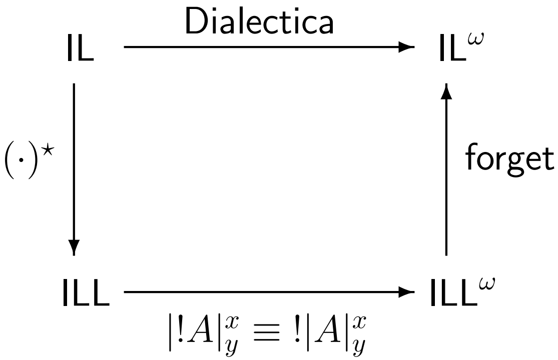
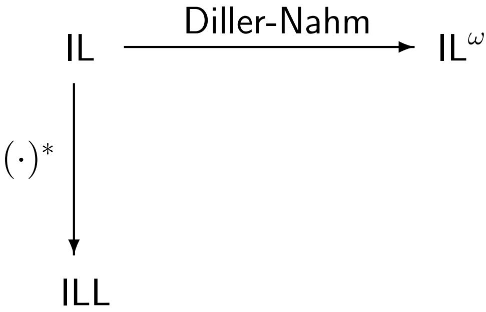


Interpretations of IL

$$\mathrm{IL} \xrightarrow{\text{Diller-Nahm}} \mathrm{IL}^{\omega}$$

Interpretations of IL



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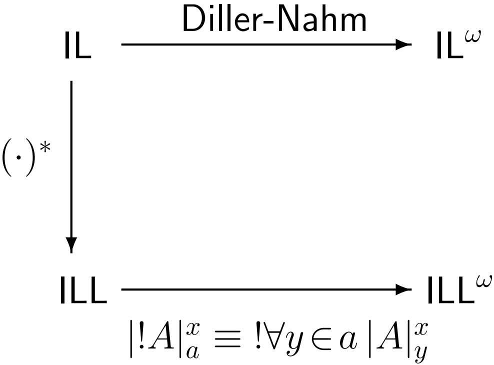

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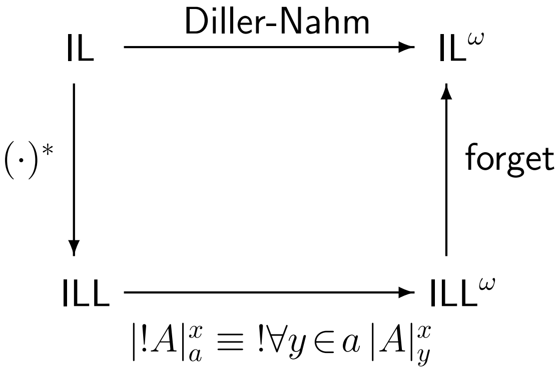
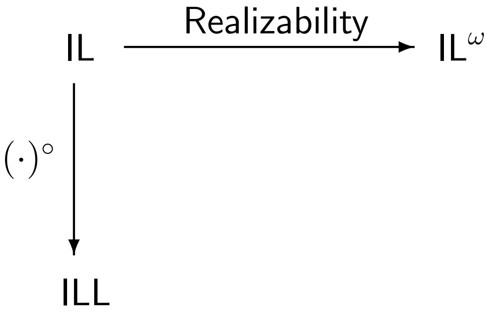


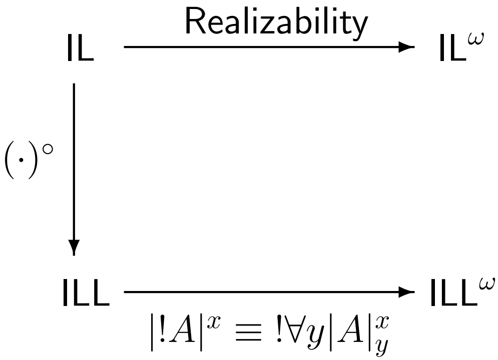
Interpretations of IL

$$\mathsf{IL} \xrightarrow{\text{Realizability}} \mathsf{IL}^{\omega}$$

Interpretations of IL



Interpretations of IL



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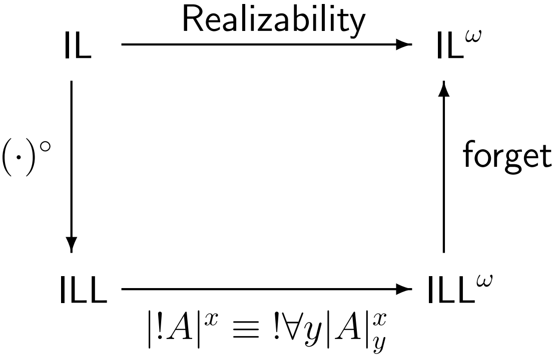

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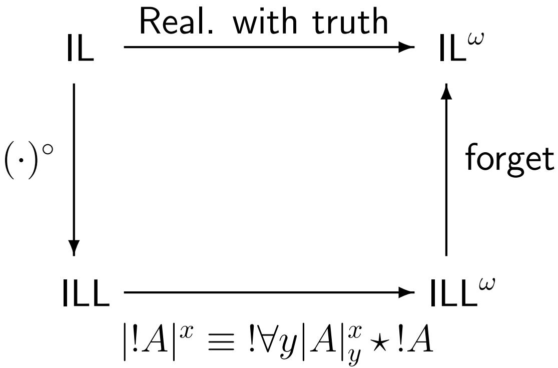
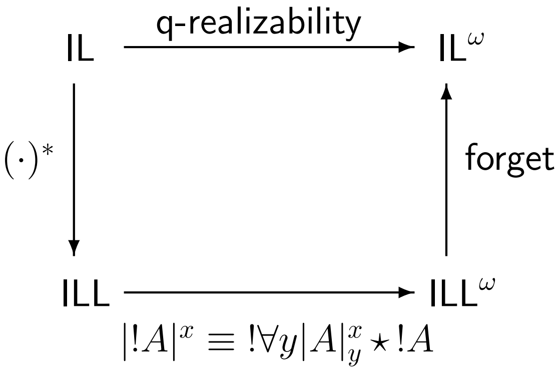


Interpretations of IL



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$$|!A|^x_a :\equiv \forall y \in a\, |A|^x_y \star\, !A$$

- New: Hybrid functional interpretation
  (exploring the fact that $!A$ not canonical in LL)

$$|!_D A|^x_y :\equiv |A|^x_y \qquad |!_r A|^x_y :\equiv \forall y |A|^x_y$$

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