Algorithms from proofs in classical arithmetic and analysis

Paulo Oliva

Queen Mary University of London

Collegium Logicum 2012

Ecole Polytechnique

16 November 2012

Theorem (Simple Theorem 1)

$\forall f^{\mathbb{N}\to\mathbb{N}} \exists n^{\mathbb{N}} (f(fn) \geq fn)$

Theorem (Simple Theorem 1)

$\forall f^{\mathbb{N}\to\mathbb{N}} \exists n^{\mathbb{N}} (f(fn) \geq fn)$

Proof.

Pick n to be a point where $f(n)$ has least value

Theorem (Simple Theorem 1)

$\forall f^{\mathbb{N}\to\mathbb{N}} \exists n^{\mathbb{N}} (f(fn) \geq fn)$

Proof.

Pick n to be a point where $f(n)$ has least value

Theorem (Effective Version)

$\forall f^{\mathbb{N}\to\mathbb{N}} \exists n \in \{f^i(0) \mid i < f0\} (f(fn) \geq fn)$

Theorem (Simple Theorem 1)

$\forall f^{\mathbb{N}\to\mathbb{N}} \exists n^{\mathbb{N}} (f(fn) \geq fn)$

Proof.

Pick n to be a point where $f(n)$ has least value

Theorem (Effective Version)

$\forall f^{\mathbb{N}\to\mathbb{N}} \exists n \in \{f^i(0) \mid i < f0\}(f(fn) \geq fn)$

Proof.

One of $n = 0$ and $n = f(0)$ and $\ldots$ and $n = f^{f0-1}(0)$ works, as the following can't happen

$$f^{f0+1}(0) < f^{f0}(0) < \ldots < f^2 0 < f0$$

How to obtain algorithms (quantitative information) from non-constructive proofs in arithmetic and analysis?

What We Assume

Decidability of atomic formulas

$$P \vee \neg P$$

What We Assume

Markov's principle

$$\neg\forall x^X A_{\mathsf{qf}} \to \exists x^X \neg A_{\mathsf{qf}}$$

What We Assume

Axiom of choice

$$\forall x^X \exists y^Y A[x,y] \to \exists f^{X\to Y} \forall x^X A[x, fx]$$

Outline

Outline

Classical Principles

Classical Principles

Law of excluded middle

$$A \vee \neg A$$

Classical Principles

Law of excluded middle

$$A \vee \neg A$$

Double negation elimination

$$\neg\neg A \to A$$

Classical Principles

Law of excluded middle

$$A \vee \neg A$$

Double negation elimination

$$\neg\neg A \to A$$

Drinker's paradox

$$\exists x (A[x] \to \forall y A[y])$$

Gödel-Gentzen Translation

Definition

$$
\begin{aligned}
P^N &:\equiv P \\
(A \wedge B)^N &:\equiv A^N \wedge B^N \\
(A \vee B)^N &:\equiv \neg\neg(A^N \vee B^N) \\
(A \to B)^N &:\equiv A^N \to B^N \\
(\exists x A)^N &:\equiv \neg\neg\exists x A^N \\
(\forall x A)^N &:\equiv \forall x A^N
\end{aligned}
$$

Gödel-Gentzen Translation

Definition

$$\begin{array}{lcl}
P^N & :\equiv & P \\
(A \wedge B)^N & :\equiv & A^N \wedge B^N \\
(A \vee B)^N & :\equiv & \neg\neg(A^N \vee B^N) \\
(A \to B)^N & :\equiv & A^N \to B^N \\
(\exists x A)^N & :\equiv & \neg\neg\exists x A^N \\
(\forall x A)^N & :\equiv & \forall x A^N
\end{array}$$

Theorem

$CL \vdash A \quad \Leftrightarrow \quad IL \vdash A^N$

Drinker's Paradox

Theorem (Simple Theorem 2)

$\forall n \exists x (fx = n \rightarrow \forall y (fy = n))$

Drinker's Paradox

Theorem (Simple Theorem 2)

$\forall n \exists x (fx = n \rightarrow \forall y (fy = n))$

Proof.

Fix n

Drinker's Paradox

Theorem (Simple Theorem 2)

$\forall n \exists x (fx = n \to \forall y (fy = n))$

Proof.

Fix n

Either $\forall y (fy = n)$, so conclusion is true, x can be anything

Drinker's Paradox

Theorem (Simple Theorem 2)

$\forall n \exists x (fx = n \to \forall y (fy = n))$

Proof.

Fix n

Either $\forall y (fy = n)$, so conclusion is true, x can be anything

Or $\exists x (fx \neq n)$, and hence $\exists x (fx = n \to B)$ for any B

(by ex-falso-quodlibet)

Drinker's Paradox

$$\forall n \exists x (fx = n \to \forall y (fy = n))$$

Drinker's Paradox

$$\forall n \exists x (fx = n \rightarrow \forall y (fy = n))$$

After negative translation

$$\forall n \neg\neg \exists x (fx = n \rightarrow \forall y (fy = n))$$

Drinker's Paradox

$$\forall n \exists x (fx = n \to \forall y (fy = n))$$

After negative translation

$$\forall n \neg\neg \exists x (fx = n \to \forall y (fy = n))$$

Prenexing

$$\forall n \neg\neg \exists x \forall y (fx = n \to fy = n) \qquad \text{[IL]}$$

Drinker's Paradox

$$\forall n \exists x (fx = n \to \forall y (fy = n))$$

After negative translation

$$\forall n \neg\neg \exists x (fx = n \to \forall y (fy = n))$$

Prenexing

$$\forall n \neg\neg \exists x \forall y (fx = n \to fy = n) \qquad [\mathsf{IL}]$$

$$\forall n \neg \forall x \exists y \neg (fx = n \to fy = n) \qquad [\mathsf{IL} + \mathsf{MP}]$$

Drinker's Paradox

$$\forall n \exists x (fx = n \to \forall y (fy = n))$$

After negative translation

$$\forall n \neg\neg \exists x (fx = n \to \forall y (fy = n))$$

Prenexing

$$\forall n \neg\neg \exists x \forall y (fx = n \to fy = n) \qquad \text{[IL]}$$

$$\forall n \neg \forall x \exists y \neg (fx = n \to fy = n) \qquad \text{[IL + MP]}$$

$$\forall n \neg {\color{red}\exists p \forall x} \neg (fx = n \to f(px) = n) \qquad \text{[AC]}$$

Drinker's Paradox

$$\forall n \exists x (fx = n \to \forall y (fy = n))$$

After negative translation

$$\forall n \neg\neg \exists x (fx = n \to \forall y (fy = n))$$

Prenexing

$$\forall n \neg\neg \exists x \forall y (fx = n \to fy = n) \qquad [\text{IL}]$$

$$\forall n \neg \forall x \exists y \neg (fx = n \to fy = n) \qquad [\text{IL} + \text{MP}]$$

$$\forall n \neg \exists p \forall x \neg (fx = n \to f(px) = n) \qquad [\text{AC}]$$

$$\forall n \forall p \exists x \neg\neg (fx = n \to f(px) = n) \qquad [\text{IL} + \text{MP}]$$

Drinker's Paradox

$$\forall n \exists x (fx = n \to \forall y (fy = n))$$

After negative translation

$$\forall n \neg\neg \exists x (fx = n \to \forall y (fy = n))$$

Prenexing

$$\forall n \neg\neg \exists x \forall y (fx = n \to fy = n) \qquad \text{[IL]}$$

$$\forall n \neg \forall x \exists y \neg (fx = n \to fy = n) \qquad \text{[IL + MP]}$$

$$\forall n \neg \exists p \forall x \neg (fx = n \to f(px) = n) \qquad \text{[AC]}$$

$$\forall n \forall p \exists x \neg\neg (fx = n \to f(px) = n) \qquad \text{[IL + MP]}$$

$$\exists \varepsilon \forall n \forall p \neg\neg (f(\varepsilon_n p) = n \to f(p(\varepsilon_n p)) = n) \qquad \text{[AC]}$$

Drinker's Paradox

Witnessing

$$\exists \varepsilon \forall n \forall p (f(\varepsilon_n p) = n \to f(p(\varepsilon_n p)) = n)$$

Drinker's Paradox

Witnessing

$$\exists \varepsilon \forall n \forall p (f(\varepsilon_n p) = n \to f(p(\varepsilon_n p)) = n)$$

p *tries to turn any witness* $\varepsilon_n p$ *into a counter-example*

Drinker's Paradox

Witnessing

$$\exists\varepsilon\forall n\forall p(f(\varepsilon_n p) = n \to f(p(\varepsilon_n p)) = n)$$

p *tries to turn any witness* $\varepsilon_n p$ *into a counter-example*

ε_n *claims that* p *can't be correct all the time*

Drinker's Paradox

Witnessing

$$\exists \varepsilon \forall n \forall p (f(\varepsilon_n p) = n \to f(p(\varepsilon_n p)) = n)$$

p tries to turn any witness $\varepsilon_n p$ into a counter-example

ε_n claims that p can't be correct all the time

Let

$$\varepsilon_n p = \begin{cases} 0 & \text{if } f(0) = n \to f(p(0)) = n \\ p(0) & \text{if } f(0) = n \wedge f(p(0)) \neq n \end{cases}$$

Drinker's Paradox

Witnessing

$$\exists \varepsilon \forall n \forall p (f(\varepsilon_n p) = n \rightarrow f(p(\varepsilon_n p)) = n)$$

p *tries to turn any witness* $\varepsilon_n p$ *into a counter-example*

ε_n *claims that* p *can't be correct all the time*

Let

$$\varepsilon_n p = \begin{cases} 0 & \text{if } f(0) = n \rightarrow f(p(0)) = n \\ p(0) & \text{if } f(0) = n \wedge f(p(0)) \neq n \end{cases}$$

Drinker's Paradox

Witnessing

$$\exists \varepsilon \forall n \forall p (f(\varepsilon_n p) = n \to f(p(\varepsilon_n p)) = n)$$

p *tries to turn any witness* $\varepsilon_n p$ *into a counter-example*

ε_n *claims that* p *can't be correct all the time*

Let

$$\varepsilon_n p = \begin{cases} 0 & \text{if } f(0) = n \to f(p(0)) = n \\ p(0) & \text{if } f(0) = n \wedge f(p(0)) \neq n \end{cases}$$

Drinker's Paradox

Witnessing

$$\exists \varepsilon \forall n \forall p (f(\varepsilon_n p) = n \rightarrow f(p(\varepsilon_n p)) = n)$$

p tries to turn any witness $\varepsilon_n p$ into a counter-example

ε_n claims that p can't be correct all the time

Let

$$\varepsilon_n p = \begin{cases} 0 & \text{if } f(0) = n \rightarrow f(p(0)) = n \\ p(0) & \text{if } f(0) = n \wedge f(p(0)) \neq n \end{cases}$$

We have, for any n and p

$$\text{if } f(\varepsilon_n p) = n \text{ then } f(p(\varepsilon_n p)) = n$$

Selection Functions

In general, given a classical theorem

$$\exists x^X \forall y^Y A(x, y)$$

Selection Functions

In general, given a classical theorem

$$\exists x^X \forall y^Y A(x, y)$$

after negative translation it becomes

$$\neg\neg\exists x^X \forall y^Y A(x, y)$$

Selection Functions

In general, given a classical theorem

$$\exists x^X \forall y^Y A(x, y)$$

after negative translation it becomes

$$\neg\neg\exists x^X \forall y^Y A(x, y)$$

after prenexation (using AC and MP)

$$\exists \varepsilon^{(X \to Y) \to X} \forall p^{X \to Y} A(\varepsilon p, p(\varepsilon p))$$

Selection Functions

In general, given a classical theorem

$$\exists x^X \forall y^Y A(x, y)$$

after negative translation it becomes

$$\neg\neg\exists x^X \forall y^Y A(x, y)$$

after prenexation (using AC and MP)

$$\exists \varepsilon^{(X \to Y) \to X} \forall p^{X \to Y} A(\varepsilon p, p(\varepsilon p))$$

We call such $\varepsilon^{(X \to Y) \to X}$ a **selection function**

Selection Functions

Mobilux LED Hand-held Magnifier				
Product #	Lens Size (mm)	Magnification	Dpt.	Price
1510-24	60	3X	12	$145.40
1510-34	75 x 50	3.5X	10	$154.80
1510-44	60	4X	16	$152.60
1510-54	60	5X	20	$154.80
1510-74	35	7X	28	$135.40
1510-104	35	10X	38	$135.40
1510-124	30	12X	48	$154.80

Selection Functions

Mobilux LED Hand-held Magnifier

Product #	Lens Size (mm)	Magnification	Dpt.	Price
1510-24	60	3X	12	$145.40
1510-34	75 x 50	3.5X	10	$154.80
1510-44	60	4X	16	$152.60
1510-54	60	5X	20	$154.80
1510-74	35	7X	28	$135.40
1510-104	35	10X	38	$135.40
1510-124	30	12X	48	$154.80

Product $\Rightarrow$ Price

Selection Functions

Mobilux LED Hand-held Magnifier				
Product #	Lens Size (mm)	Magnification	Dpt.	Price
1510-24	60	3X	12	$145.40
1510-34	75 x 50	3.5X	10	$154.80
1510-44	60	4X	16	$152.60
1510-54	60	5X	20	$154.80
1510-74	35	7X	28	$135.40
1510-104	35	10X	38	$135.40
1510-124	30	12X	48	$154.80

$$(\textbf{Product} \Rightarrow \textbf{Price}) \Rightarrow \textbf{Product}$$

Selection Functions

Mobilux LED Hand-held Magnifier				
Product #	Lens Size (mm)	Magnification	Dpt.	Price
1510-24	60	3X	12	$145.40
1510-34	75 x 50	3.5X	10	$154.80
1510-44	60	4X	16	$152.60
1510-54	60	5X	20	$154.80
1510-74	35	7X	28	$135.40
1510-104	35	10X	38	$135.40
1510-124	30	12X	48	$154.80

Move $\Rightarrow$ Outcome

Selection Functions

Mobilux LED Hand-held Magnifier

Product #	Lens Size (mm)	Magnification	Dpt.	Price
1510-24	60	3X	12	$145.40
1510-34	75 x 50	3.5X	10	$154.80
1510-44	60	4X	16	$152.60
1510-54	60	5X	20	$154.80
1510-74	35	7X	28	$135.40
1510-104	35	10X	38	$135.40
1510-124	30	12X	48	$154.80

$$(\text{Move} \Rightarrow \text{Outcome}) \Rightarrow \text{Move}$$

More on Selection Functions and Sequential Games

M. Escardó and P. Oliva
Selection functions, bar recursion and backward induction
MSCS, 20(2):127-168, 2010

M. Escardó and P. Oliva
What sequential games, the Tychnoff theorem and the double-negation shift have in common
ACM SIGPLAN MSFP, ACM Press 2010

M. Escardó and P. Oliva
Sequential games and optimal strategies
Proceedings of the Royal Society A, 2011

Outline

Infinite Pigeonhole Principle

Theorem (Simple Theorem 3)

For every number of colours $n \colon \mathbb{N}$ *and colouring* $f \colon \mathbb{N} \to n$ *one colour must be used infinitely often, i.e.*

$$\exists i < n \forall k \exists j (j \geq k \wedge f(j) = i)$$

Proof.

Assume, for the sake of a contradiction that

$$\forall i < n \exists k \forall j \geq k (f(j) \neq i).$$

By Π_1-bounded collection there exists an M such that

$$\forall i < n \exists k \leq M \forall j \geq k (f(j) \neq i).$$

In particular $\forall i < n \forall j \geq M (f(j) \neq i)$, which is clearly false.

Infinite Pigeonhole Principle

For every $n \colon \mathbb{N}$ and $f \colon \mathbb{N} \to n$

$$\exists i < n \forall k \exists j (j \geq k \wedge f(j) = i)$$

Infinite Pigeonhole Principle

For every $n\colon \mathbb{N}$ and $f : \mathbb{N} \to n$

$$\exists i < n \forall k \exists j (j \geq k \wedge f(j) = i)$$

After negative translation

$$\neg\neg\exists i < n \forall k \exists j (j \geq k \wedge f(j) = i)$$

Infinite Pigeonhole Principle

For every $n\colon \mathbb{N}$ and $f\colon \mathbb{N} \to n$

$$\exists i < n \forall k \exists j (j \geq k \wedge f(j) = i)$$

After negative translation

$$\neg\neg\exists i < n \forall k \exists j (j \geq k \wedge f(j) = i)$$

Prenexing

$$\neg\neg\exists i < n \exists p \forall k (pk \geq k \wedge f(pk) = i) \qquad \text{[AC]}$$

Infinite Pigeonhole Principle

For every $n\colon \mathbb{N}$ and $f\colon \mathbb{N} \to n$

$$\exists i < n \forall k \exists j (j \geq k \wedge f(j) = i)$$

After negative translation

$$\neg\neg\exists i < n \forall k \exists j (j \geq k \wedge f(j) = i)$$

Prenexing

$$\neg\neg\exists i < n \exists p \forall k (pk \geq k \wedge f(pk) = i) \qquad [\mathsf{AC}]$$

$$\neg\forall i < n \forall p \exists k \neg(pk \geq k \wedge f(pk) = i) \qquad [\mathsf{IL} + \mathsf{MP}]$$

Infinite Pigeonhole Principle

For every $n\colon \mathbb{N}$ and $f\colon \mathbb{N} \to n$

$$\exists i < n \forall k \exists j (j \geq k \wedge f(j) = i)$$

After negative translation

$$\neg\neg\exists i < n \forall k \exists j (j \geq k \wedge f(j) = i)$$

Prenexing

$$\neg\neg\exists i < n \exists p \forall k (pk \geq k \wedge f(pk) = i) \qquad [\text{AC}]$$

$$\neg\forall i < n \forall p \exists k \neg (pk \geq k \wedge f(pk) = i) \qquad [\text{IL} + \text{MP}]$$

$$\neg\exists\varepsilon_{(\cdot)} \forall i < n \forall p \neg (p(\varepsilon_i p) \geq \varepsilon_i p \wedge f(p(\varepsilon_i p)) = i) \qquad [\text{AC}]$$

Infinite Pigeonhole Principle

For every $n : \mathbb{N}$ and $f : \mathbb{N} \to n$

$$\exists i < n \forall k \exists j (j \geq k \wedge f(j) = i)$$

After negative translation

$$\neg\neg\exists i < n \forall k \exists j (j \geq k \wedge f(j) = i)$$

Prenexing

$$\neg\neg\exists i < n \exists p \forall k (pk \geq k \wedge f(pk) = i) \qquad [\mathsf{AC}]$$

$$\neg\forall i < n \forall p \exists k \neg (pk \geq k \wedge f(pk) = i) \qquad [\mathsf{IL} + \mathsf{MP}]$$

$$\neg\exists \varepsilon_{(\cdot)} \forall i < n \forall p \neg (p(\varepsilon_i p) \geq \varepsilon_i p \wedge f(p(\varepsilon_i p)) = i) \qquad [\mathsf{AC}]$$

$$\forall \varepsilon_{(\cdot)} \exists i < n \exists p (p(\varepsilon_i p) \geq \varepsilon_i p \wedge f(p(\varepsilon_i p)) = i) \qquad [\mathsf{MP}]$$

Infinite Pigeonhole Principle

For every $n : \mathbb{N}$ and $f : \mathbb{N} \to n$

$$\exists i < n \forall k \exists j (j \geq k \wedge f(j) = i)$$

After negative translation

$$\neg\neg\exists i < n \forall k \exists j (j \geq k \wedge f(j) = i)$$

Prenexing

$$\neg\neg\exists i < n \exists p \forall k (pk \geq k \wedge f(pk) = i) \qquad [\text{AC}]$$

$$\neg\forall i < n \forall p \exists k \neg(pk \geq k \wedge f(pk) = i) \qquad [\text{IL} + \text{MP}]$$

$$\neg\exists \varepsilon_{(\cdot)} \forall i < n \forall p \neg(p(\varepsilon_i p) \geq \varepsilon_i p \wedge f(p(\varepsilon_i p)) = i) \qquad [\text{AC}]$$

$$\forall \varepsilon_{(\cdot)} \exists i < n \exists p (p(\varepsilon_i p) \geq \varepsilon_i p \wedge f(p(\varepsilon_i p)) = i) \qquad [\text{MP}]$$

Let us consider $n = 2$ (two colours)

Let us consider $n = 2$ (two colours)

For every $f \colon \mathbb{N} \to 2$ and $\varepsilon_0, \varepsilon_1 \colon (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$

there exists a $p \colon \mathbb{N} \to \mathbb{N}$ such that either

$$p(\varepsilon_0 p) \geq \varepsilon_0 p \wedge f(p(\varepsilon_0 p)) = 0$$

or

$$p(\varepsilon_1 p) \geq \varepsilon_1 p \wedge f(p(\varepsilon_1 p)) = 1$$

Let us consider $n = 2$ (two colours)

For every $f\colon \mathbb{N} \to 2$ and $\varepsilon_0, \varepsilon_1\colon (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$

there exists a $p\colon \mathbb{N} \to \mathbb{N}$ such that either

$$p(\varepsilon_0 p) \geq \varepsilon_0 p \wedge f(p(\varepsilon_0 p)) = 0$$

or

$$p(\varepsilon_1 p) \geq \varepsilon_1 p \wedge f(p(\varepsilon_1 p)) = 1$$

Let

$$p_0(x) = \max(x, \varepsilon_1(\lambda y. \max(x, y)))$$
$$p_1(y) = \max(\varepsilon_0 p_0, y)$$

Let us consider $n = 2$ (two colours)

For every $f : \mathbb{N} \to 2$ and $\varepsilon_0, \varepsilon_1 : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$

there exists a $p : \mathbb{N} \to \mathbb{N}$ such that either

$$p(\varepsilon_0 p) \geq \varepsilon_0 p \wedge f(p(\varepsilon_0 p)) = 0$$

or

$$p(\varepsilon_1 p) \geq \varepsilon_1 p \wedge f(p(\varepsilon_1 p)) = 1$$

Let

$$p_0(x) = \max(x, \varepsilon_1(\lambda y. \max(x, y)))$$

$$p_1(y) = \max(\varepsilon_0 p_0, y)$$

Clearly $p_0(x) \geq x$ and $p_1(y) \geq y$.

Let us consider $n = 2$ (two colours)

For every $f\colon \mathbb{N} \to 2$ and $\varepsilon_0, \varepsilon_1\colon (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$

there exists a $p\colon \mathbb{N} \to \mathbb{N}$ such that either

$$p(\varepsilon_0 p) \geq \varepsilon_0 p \wedge f(p(\varepsilon_0 p)) = 0$$

or

$$p(\varepsilon_1 p) \geq \varepsilon_1 p \wedge f(p(\varepsilon_1 p)) = 1$$

Let

$$p_0(x) = \max(x, \varepsilon_1(\lambda y.\max(x, y)))$$

$$p_1(y) = \max(\varepsilon_0 p_0, y)$$

Clearly $p_0(x) \geq x$ and $p_1(y) \geq y$.

Claim. Either $f(p_0(\varepsilon_0 p_0)) = 0$ or $f(p_1(\varepsilon_1 p_1)) = 1$

Given

$$p_0(x) = \max(x, \varepsilon_1(\lambda y. \max(x, y)))$$
$$p_1(y) = \max(\varepsilon_0 p_0, y)$$

Lemma

Either $f(p_0(\varepsilon_0 p_0)) = 0$ *or* $f(p_1(\varepsilon_1 p_1)) = 1$

Given

$$p_0(x) = \max(x, \varepsilon_1(\lambda y.\max(x, y)))$$

$$p_1(y) = \max(\varepsilon_0 p_0, y)$$

Lemma

Either $f(p_0(\varepsilon_0 p_0)) = 0$ *or* $f(p_1(\varepsilon_1 p_1)) = 1$

Proof.

Given

$$p_0(x) = \max(x, \varepsilon_1(\lambda y. \max(x, y)))$$
$$p_1(y) = \max(\varepsilon_0 p_0, y)$$

Lemma

Either $f(p_0(\varepsilon_0 p_0)) = 0$ *or* $f(p_1(\varepsilon_1 p_1)) = 1$

Proof.

Note that

$$p_1(\varepsilon_1 p_1) = \max(\varepsilon_0 p_0, \varepsilon_1 p_1)$$

Given

$$p_0(x) = \max(x, \varepsilon_1(\lambda y. \max(x, y)))$$
$$p_1(y) = \max(\varepsilon_0 p_0, y)$$

Lemma

Either $f(p_0(\varepsilon_0 p_0)) = 0$ *or* $f(p_1(\varepsilon_1 p_1)) = 1$

Proof.

Note that

$$\begin{aligned} p_1(\varepsilon_1 p_1) &= \max(\varepsilon_0 p_0, \varepsilon_1 p_1) \\ p_0(\varepsilon_0 p_0) &= \max(\varepsilon_0 p_0, \varepsilon_1 p_1) \end{aligned}$$

because $\varepsilon_1(\lambda y. \max(\varepsilon_0 p_0, y)) = \varepsilon_1 p_1$

Given

$$p_0(x) = \max(x, \varepsilon_1(\lambda y.\max(x, y)))$$
$$p_1(y) = \max(\varepsilon_0 p_0, y)$$

Lemma

Either $f(p_0(\varepsilon_0 p_0)) = 0$ *or* $f(p_1(\varepsilon_1 p_1)) = 1$

Proof.

Note that

$$p_1(\varepsilon_1 p_1) = \max(\varepsilon_0 p_0, \varepsilon_1 p_1)$$
$$p_0(\varepsilon_0 p_0) = \max(\varepsilon_0 p_0, \varepsilon_1 p_1)$$

because $\varepsilon_1(\lambda y.\max(\varepsilon_0 p_0, y)) = \varepsilon_1 p_1$

Check colour $f(\max(\varepsilon_0 p_0, \varepsilon_1 p_1))$

If 0 then $f(p_0(\varepsilon_0 p_0)) = 0$ else $f(p_1(\varepsilon_1 p_1)) = 1$

Product of Selection Functions

Given selection functions

$$\begin{aligned} \varepsilon_0 &: (X \to R) \to X \\ \varepsilon_1 &: (Y \to R) \to Y \end{aligned}$$

we have built a single selection function $(\varepsilon_0 \otimes \varepsilon_1)$ of type

$$(X \times Y \to R) \to X \times Y$$

Product of Selection Functions

Given selection functions

$$\begin{aligned} \varepsilon_0 &: (X \to R) \to X \\ \varepsilon_1 &: (Y \to R) \to Y \end{aligned}$$

we have built a single selection function $(\varepsilon_0 \otimes \varepsilon_1)$ of type

$$(X \times Y \to R) \to X \times Y$$

as

$$(\varepsilon_0 \otimes \varepsilon_1)(q^{X \times Y \to R}) = (\varepsilon_0 p_0, \varepsilon_1 p_1)$$

where

$$\begin{aligned} p_0(x) &\stackrel{R}{=} q(x, \varepsilon_1(\lambda y. q(x, y))) \\ p_1(y) &\stackrel{R}{=} q(\varepsilon_0 p_0, y) \end{aligned}$$

Product of Selection Functions – Theorem

Definition (Escardó/O.'2008)

Given a family of selection functions $\varepsilon_i \colon (X_i \to R) \to X_i$ we define their iterated product as

$$\left(\bigotimes_{i=k}^{\infty} \varepsilon_i \right) = \varepsilon_k \otimes \left(\bigotimes_{i=k+1}^{\infty} \varepsilon_i \right)$$

Product of Selection Functions – Theorem

Definition (Escardó/O.'2008)

Given a family of selection functions $\varepsilon_i \colon (X_i \to R) \to X_i$ we define their iterated product as

$$\left(\bigotimes_{i=k}^{\infty} \varepsilon_i\right) = \varepsilon_k \otimes \left(\bigotimes_{i=k+1}^{\infty} \varepsilon_i\right)$$

Theorem (Escardó/O.'2008)

Given $\varepsilon_i \colon (X_i \to R) \to X_i$ *and* $q \colon \Pi_i X_i \to R$ *let* $\alpha = (\bigotimes_i \varepsilon_i)(q)$. *There exists* $p_i \colon X_i \to R$ *such that*

$$\alpha(i) \stackrel{X_i}{=} \varepsilon_i p_i$$

$$q\alpha \stackrel{R}{=} p_i(\varepsilon_i p_i)$$

How to Use This

Theorem (Escardó/O.'2008)

Given $\varepsilon_i \colon (X_i \to R) \to X_i$ *and* $q \colon \Pi_i X_i \to R$ *let* $\alpha = (\bigotimes_i \varepsilon_i)(q)$. *There exists* $p_i \colon X_i \to R$ *such that*

$$\begin{aligned} \alpha(i) &\stackrel{X_i}{=} \varepsilon_i p_i \\ q\alpha &\stackrel{R}{=} p_i(\varepsilon_i p_i) \end{aligned}$$

How to Use This

Theorem (Escardó/O.'2008)

Given $\varepsilon_i \colon (X_i \to R) \to X_i$ *and* $q \colon \Pi_i X_i \to R$ *let* $\alpha = (\bigotimes_i \varepsilon_i)(q)$. *There exists* $p_i \colon X_i \to R$ *such that*

$$\begin{aligned} \alpha(i) &\stackrel{X_i}{=} \varepsilon_i p_i \\ q\alpha &\stackrel{R}{=} p_i(\varepsilon_i p_i) \end{aligned}$$

In order to produce a witness α for

$$\forall q^{\Pi_i X_i \to R} \exists \alpha^{\Pi_i X_i} \forall i A_i(\alpha(i), q\alpha)$$

How to Use This

Theorem (Escardó/O.'2008)

Given $\varepsilon_i \colon (X_i \to R) \to X_i$ *and* $q \colon \Pi_i X_i \to R$ *let* $\alpha = (\bigotimes_i \varepsilon_i)(q)$. *There exists* $p_i \colon X_i \to R$ *such that*

$$\begin{array}{lcl} \alpha(i) & \stackrel{X_i}{=} & \varepsilon_i p_i \\ q\alpha & \stackrel{R}{=} & p_i(\varepsilon_i p_i) \end{array}$$

In order to produce a witness α for

$$\forall q^{\Pi_i X_i \to R} \exists \alpha^{\Pi_i X_i} \forall i A_i(\alpha(i), q\alpha)$$

it is enough to produce selection functions $\varepsilon_{(\cdot)}$ witnessing

$$\exists \varepsilon_{(\cdot)} \forall p, i A_i(\varepsilon_i p, p(\varepsilon_i p))$$

Outline

Inverse of a Function

Lemma (Simple Lemma)

For any $H\colon X \to \mathbb{N}$ *there exists* $\alpha\colon \mathbb{N} \to X$ *such that*

$$H(\alpha k) = k \qquad \textit{whenever} \quad k \in \mathrm{img}(H)$$

Inverse of a Function

Lemma (Simple Lemma)

For any $H: X \to \mathbb{N}$ *there exists* $\alpha: \mathbb{N} \to X$ *such that*

$$H(\alpha k) = k \qquad \textit{whenever} \quad k \in \mathrm{img}(H)$$

Proof.

From **logical axiom**

$$\forall k(\exists x(Hx = k) \to \exists x'(Hx' = k))$$

Inverse of a Function

Lemma (Simple Lemma)

For any $H\colon X \to \mathbb{N}$ *there exists* $\alpha\colon \mathbb{N} \to X$ *such that*

$$H(\alpha k) = k \qquad \textit{whenever} \quad k \in \mathrm{img}(H)$$

Proof.

From **logical axiom**

$$\forall k(\exists x(Hx = k) \to \exists x'(Hx' = k))$$

Inverse of a Function

Lemma (Simple Lemma)

For any $H\colon X \to \mathbb{N}$ *there exists* $\alpha\colon \mathbb{N} \to X$ *such that*

$$H(\alpha k) = k \qquad \textit{whenever} \quad k \in \mathrm{img}(H)$$

Proof.

From **logical axiom**

$$\forall k(\exists x(Hx = k) \to \exists x'(Hx' = k))$$

prenex x' (drinker's paradox)

$$\forall k \exists x'(\exists x(Hx = k) \to Hx' = k)$$

Inverse of a Function

Lemma (Simple Lemma)

For any $H\colon X \to \mathbb{N}$ *there exists* $\alpha\colon \mathbb{N} \to X$ *such that*

$$H(\alpha k) = k \qquad \textit{whenever} \quad k \in \mathrm{img}(H)$$

Proof.

From **logical axiom**

$$\forall k(\exists x(Hx = k) \to \exists x'(Hx' = k))$$

prenex x' (drinker's paradox)

$$\forall k \exists x'(\exists x(Hx = k) \to Hx' = k)$$

Inverse of a Function

Lemma (Simple Lemma)

For any $H\colon X \to \mathbb{N}$ *there exists* $\alpha\colon \mathbb{N} \to X$ *such that*

$$H(\alpha k) = k \qquad \textit{whenever} \qquad k \in \mathrm{img}(H)$$

Proof.

From **logical axiom**

$$\forall k(\exists x(Hx = k) \to \exists x'(Hx' = k))$$

prenex x' (drinker's paradox)

$$\forall k \exists x'(\exists x(Hx = k) \to Hx' = k)$$

and invoke the **axiom of (countable) choice**

$$\exists \alpha \forall k(\exists x(Hx = k) \to H(\alpha k) = k)$$

No Injection from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$

Theorem (Simple Theorem 4)

For any $H\colon (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ *there exist* $f, g\colon \mathbb{N} \to \mathbb{N}$ *such that*

$$f \neq g \quad \textit{and} \quad H(f) \stackrel{\mathbb{N}}{=} H(g)$$

No Injection from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$

Theorem (Simple Theorem 4)

For any $H : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ *there exist* $f, g : \mathbb{N} \to \mathbb{N}$ *such that*

$$f \neq g \quad \textit{and} \quad H(f) \overset{\mathbb{N}}{=} H(g)$$

Proof.

Let $\alpha^{\mathbb{N} \to \mathbb{N}^{\mathbb{N}}}$ be some inverse of H, i.e. for all k

$$(*) \quad H(\alpha(k)) = k \qquad \text{if } k \in \text{img}(H)$$

(using **classical logic** and **countable choice**)

No Injection from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$

Theorem (Simple Theorem 4)

For any $H : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ *there exist* $f, g : \mathbb{N} \to \mathbb{N}$ *such that*

$$f \neq g \quad \textit{and} \quad H(f) \overset{\mathbb{N}}{=} H(g)$$

Proof.

Let $\alpha^{\mathbb{N} \to \mathbb{N}^{\mathbb{N}}}$ be some inverse of H, i.e. for all k

$$(*) \quad H(\alpha(k)) = k \qquad \text{if} \;\; k \in \text{img}(H)$$

(using **classical logic** and **countable choice**)

Let $f_\alpha = \lambda n.\alpha(n)(n) + 1$ and $g_\alpha = \alpha(k_\alpha)$ where $k_\alpha = H(f_\alpha)$

No Injection from $\mathbb{N}^\mathbb{N}$ to $\mathbb{N}$

Theorem (Simple Theorem 4)

For any $H : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ *there exist* $f, g : \mathbb{N} \to \mathbb{N}$ *such that*

$$f \neq g \quad \textit{and} \quad H(f) \overset{\mathbb{N}}{=} H(g)$$

Proof.

Let $\alpha^{\mathbb{N} \to \mathbb{N}^\mathbb{N}}$ be some inverse of H, i.e. for all k

$$(*) \quad H(\alpha(k)) = k \qquad \text{if } \ k \in \text{img}(H)$$

(using **classical logic** and **countable choice**)

Let $f_\alpha = \lambda n.\alpha(n)(n) + 1$ and $g_\alpha = \alpha(k_\alpha)$ where $k_\alpha = H(f_\alpha)$

Clearly $f_\alpha(k_\alpha) \neq g_\alpha(k_\alpha)$

No Injection from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$

Theorem (Simple Theorem 4)

For any $H : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ *there exist* $f, g : \mathbb{N} \to \mathbb{N}$ *such that*

$$f \neq g \quad \textit{and} \quad H(f) \overset{\mathbb{N}}{=} H(g)$$

Proof.

Let $\alpha^{\mathbb{N} \to \mathbb{N}^{\mathbb{N}}}$ be some inverse of H, i.e. for all k

$$(*) \quad H(\alpha(k)) = k \qquad \text{if } \; k \in \mathrm{img}(H)$$

(using **classical logic** and **countable choice**)

Let $f_\alpha = \lambda n.\alpha(n)(n) + 1$ and $g_\alpha = \alpha(k_\alpha)$ where $k_\alpha = H(f_\alpha)$

Clearly $f_\alpha(k_\alpha) \neq g_\alpha(k_\alpha)$ and $H(f_\alpha) = k_\alpha \overset{(*)}{=} H(g_\alpha)$

No Injection from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$

Theorem (Simple Theorem 4)

For any $H : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ *there exist* $f, g : \mathbb{N} \to \mathbb{N}$ *such that*

$$f \neq g \quad \textit{and} \quad H(f) \stackrel{\mathbb{N}}{=} H(g)$$

Proof.

Let $\alpha^{\mathbb{N} \to \mathbb{N}^{\mathbb{N}}}$ be some inverse of H, i.e. for all f and k

$$(*) \quad H(\alpha(k)) = k \qquad \text{if} \;\; H(f) = k$$

(using **classical logic** and **countable choice**)

Let $f_\alpha = \lambda n.\alpha(n)(n) + 1$ and $g_\alpha = \alpha(k_\alpha)$ where $k_\alpha = H(f_\alpha)$

Clearly $f_\alpha(k_\alpha) \neq g_\alpha(k_\alpha)$ and $H(f_\alpha) = k_\alpha \stackrel{(*)}{=} H(g_\alpha)$

No Injection from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$

Theorem (Simple Theorem 4)

For any $H : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ there exist $f, g : \mathbb{N} \to \mathbb{N}$ such that

$$f \neq g \quad \textit{and} \quad H(f) \stackrel{\mathbb{N}}{=} H(g)$$

Proof.

Let $\alpha^{\mathbb{N} \to \mathbb{N}^{\mathbb{N}}}$ be some inverse of H, i.e. for all f ~~and k~~

$$(*) \quad H(\alpha(k_\alpha)) = k_\alpha \qquad \text{if} \;\; H(f) = k_\alpha$$

(using **classical logic** and **countable choice**)

Let $f_\alpha = \lambda n.\alpha(n)(n) + 1$ and $g_\alpha = \alpha(k_\alpha)$ where $k_\alpha = H(f_\alpha)$

Clearly $f_\alpha(k_\alpha) \neq g_\alpha(k_\alpha)$ and $H(f_\alpha) = k_\alpha \stackrel{(*)}{=} H(g_\alpha)$

No Injection from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$

Theorem (Simple Theorem 4)

For any $H\colon (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ there exist $f, g\colon \mathbb{N} \to \mathbb{N}$ such that

$$f \neq g \quad \textit{and} \quad H(f) \overset{\mathbb{N}}{=} H(g)$$

Proof.

Let $\alpha^{\mathbb{N} \to \mathbb{N}^{\mathbb{N}}}$ be some inverse of H, i.e. for all f ~~and k~~

$$(*) \quad H(\alpha(k_\alpha)) = k_\alpha \qquad \text{if} \;\; H(f) = k_\alpha$$

(using **classical logic** and **countable choice**)

Let $f_\alpha = \lambda n.\alpha(n)(n) + 1$ and $g_\alpha = \alpha(k_\alpha)$ where $k_\alpha = H(f_\alpha)$

Clearly $f_\alpha(k_\alpha) \neq g_\alpha(k_\alpha)$ and $H(f_\alpha) = k_\alpha \overset{(*)}{=} H(g_\alpha)$

No Injection from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$

Theorem (Simple Theorem 4)

For any $H : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ *there exist* $f, g : \mathbb{N} \to \mathbb{N}$ *such that*

$$f \neq g \quad \textit{and} \quad H(f) \stackrel{\mathbb{N}}{=} H(g)$$

Proof.

Let $\alpha^{\mathbb{N} \to \mathbb{N}^{\mathbb{N}}}$ be some inverse of H, i.e. ~~for all f and k~~

$$(*) \quad H(\alpha(k_\alpha)) = k_\alpha \qquad \text{if } \; H(f_\alpha) = k_\alpha$$

(using **classical logic** and **countable choice**)

Let $f_\alpha = \lambda n.\alpha(n)(n) + 1$ and $g_\alpha = \alpha(k_\alpha)$ where $k_\alpha = H(f_\alpha)$

Clearly $f_\alpha(k_\alpha) \neq g_\alpha(k_\alpha)$ and $H(f_\alpha) = k_\alpha \stackrel{(*)}{=} H(g_\alpha)$

No Injection from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$

Construct approximation to inverse of H, i.e. $\alpha^{\mathbb{N}\to\mathbb{N}^{\mathbb{N}}}$ s.t.

$$\forall k \leq H(f_\alpha) \left(\underbrace{H(f_\alpha) = k \;\to\; H(\alpha(k)) = k}_{A_k(\alpha(k), f_\alpha)} \right)$$

No Injection from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$

Construct approximation to inverse of H, i.e. $\alpha^{\mathbb{N}\to\mathbb{N}^{\mathbb{N}}}$ s.t.

$$\forall k \leq H(f_\alpha) \left(\underbrace{H(f_\alpha) = k \ \to \ H(\alpha(k)) = k}_{A_k(\alpha(k), f_\alpha)} \right)$$

Enough to produce ε_k's such that for all p and k

$$\underbrace{H(p(\varepsilon_k p)) = k \ \to \ H(\varepsilon_k p) = k}_{A_k(\varepsilon_k p, p(\varepsilon_k p))}$$

No Injection from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$

Construct approximation to inverse of H, i.e. $\alpha^{\mathbb{N} \to \mathbb{N}^{\mathbb{N}}}$ s.t.

$$\forall k \leq H(f_\alpha) \left(\underbrace{H(f_\alpha) = k \;\to\; H(\alpha(k)) = k}_{A_k(\alpha(k), f_\alpha)} \right)$$

Enough to produce ε_k's such that for all p and k

$$\underbrace{H(p(\varepsilon_k p)) = k \;\to\; H(\varepsilon_k p) = k}_{A_k(\varepsilon_k p, p(\varepsilon_k p))}$$

We have built these when solving the drinker paradox!

No Injection from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$

Let ε_i as in drinker's paradox and $f_\alpha := \lambda n.\alpha(n)(n) + 1$

Theorem

Fix $H : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$. *Let* $q\alpha \overset{\mathbb{N}^{\mathbb{N}}}{=} f_\alpha$ *and* $\psi\alpha \overset{\mathbb{N}}{=} H(f_\alpha)$. *Define*

$$\alpha = \left(\bigotimes_{\langle\,\rangle}^{\psi} \varepsilon \right)(q)$$

and $f = f_\alpha$ *and* $g = \alpha(\psi\alpha)$. *Then*

$$Hf = Hg \qquad \text{and} \qquad f(\psi\alpha) \neq g(\psi\alpha)$$