Gödel's *dialectica* interpretation

(classical logic, arithmetic and analysis)

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31 August 2012

Theorem

$$\forall f^{\mathbb{N}\to\mathbb{N}} \exists n^{\mathbb{N}} (fn \leq f(fn))$$

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Proof.

Pick n to be a point where $f(n)$ has least value

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Proof.

One of $n = 0$ and $n = f(0)$ and $\ldots$ and $n = f^{f0-1}(0)$ works, as the following can't happen

$$f0 > f^2 0 > \ldots > f^{f0}(0) > f^{f0+1}(0)$$

Outline

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Inverse of a Function

Theorem

For any $H \colon X \to \mathbb{N}$ *there exists* $\alpha \colon \mathbb{N} \to X$ *such that*

$$H(\alpha k) = k \qquad \textit{whenever} \quad k \in \mathrm{img}(H)$$

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From **logical axiom**

$$\forall k(\exists x(Hx = k) \to \exists x'(Hx' = k))$$

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and invoke the **axiom of (countable) choice**

$$\exists \alpha \forall k(\exists x(Hx = k) \to H(\alpha k) = k)$$

No Injection from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$

Corollary

For any $H\colon (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ *there exist* $f, g\colon \mathbb{N} \to \mathbb{N}$ *such that*

$$f \neq g \quad \textit{and} \quad H(f) \stackrel{\mathbb{N}}{=} H(g)$$

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(using **classical logic** and **countable choice**)

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(using **classical logic** and **countable choice**)

Let $f_\alpha = \lambda n.\alpha(n)(n) + 1$ and $g_\alpha = \alpha(k_\alpha)$ where $k_\alpha = H(f_\alpha)$

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Let $f_\alpha = \lambda n.\alpha(n)(n) + 1$ and $g_\alpha = \alpha(k_\alpha)$ where $k_\alpha = H(f_\alpha)$

Clearly $f_\alpha(k_\alpha) \neq g_\alpha(k_\alpha)$ and $H(f_\alpha) = k_\alpha \stackrel{(*)}{=} H(g_\alpha)$

Drinker's Paradox

How to "witness" a theorem like this:

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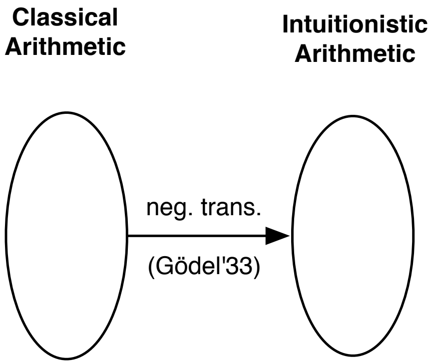
Q: *What does it mean to computationally interpret this?*


Classical Arithmetic
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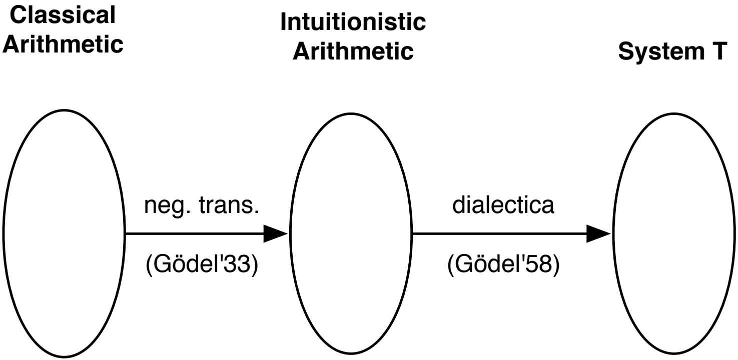
neg. trans.
(Gödel'33)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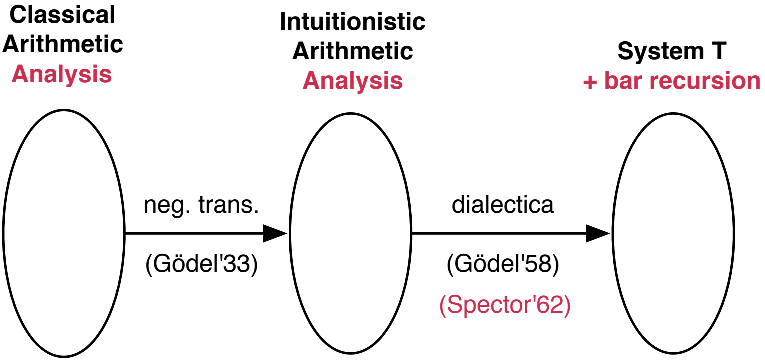
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Map every formula to the ∃∀-fragment. For instance:

$$\exists x \forall y P(x, y) \qquad \mapsto \quad \exists x\ \forall y\ P(x, y)$$

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\neg\neg \exists x \forall y P(x,y) & \mapsto & \exists \varepsilon\ \forall p\ \neg\neg P(\varepsilon p, p(\varepsilon p))
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Gödel's *dialectica* Interpretation

Can think of the mapping

$$A \qquad \mapsto \qquad \exists x \forall y A_D(x, y)$$

as associating a **set of functionals** to each formula

$$A \qquad \mapsto \qquad W_A \equiv \{ f : \forall y A_D(f, y) \}$$

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Theorem (Soundness – Intuitionistic Version)

If A is HA-provable then W_A is non-empty.

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Theorem (Soundness – Intuitionistic Version)

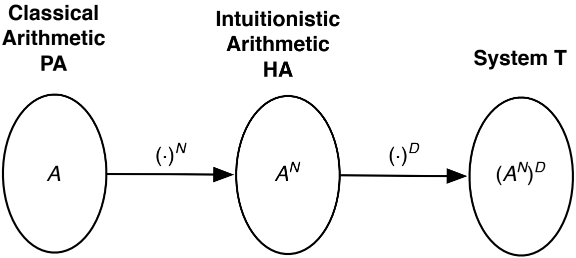
If A is HA-provable then W_A is non-empty. That is, if

(1) A is provable in Heyting arithmetic

then

(2) $A_D(t, y)$ is provable in a quantifier-free calculus T,

for some term $t \in$ T.



Theorem (Soundness – Classical Version)

Assume A^N *interpreted as* $\exists x \forall y A_D^N(x,y)$. *If*

(1) A *is provable in Peano arithmetic*

then

(2) $A_D^N(t,y)$ *is provable in the quantifier-free calculus T,*

for some term $t \in T$.

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We can prove (classically)

$$\forall i \exists x \forall y (Q_i(y) \rightarrow Q_i(x))$$

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Whose dialectica interpretation is

$$\exists \varepsilon_{(\cdot)} \forall i, p (Q_i(p(\varepsilon_i p)) \to Q_i(\varepsilon_i p))$$

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$$\exists \varepsilon_{(\cdot)} \forall i, p (Q_i(p(\varepsilon_i p)) \to Q_i(\varepsilon_i p))$$

which has witness

$$\varepsilon_i p = \begin{cases} 0 & \text{if } \neg Q_i(p0) \\ p0 & \text{if } \quad Q_i(p0) \end{cases}$$

Outline

Selection Functions

In general

$$\neg\neg\exists x^X \forall r^R Q(x, r) \quad \mapsto \quad \exists \varepsilon^{(X \to R) \to X} \forall p^{X \to R} Q(\varepsilon p, p(\varepsilon p))$$

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Let

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Let

$$J_R X \equiv (X \to R) \to X$$

We think of the objects of type $J_R X$ as **selection functions**

Consider:

- $R = \mathbb{B}$
- think of $p\colon X \to \mathbb{B}$ as a predicate over X
- $\varepsilon\colon J_R X$ picks some $\varepsilon p = x \in X$ given a subset $p \subseteq X$

Binary Product of Selection Functions

Definition

Given $\varepsilon\colon J_R X$ and $\delta\colon J_R Y$ define their product

$$(\varepsilon \otimes \delta)\colon J_R(X \times Y)$$

as

$$(\varepsilon \otimes \delta)(q^{X \times Y \to R}) \stackrel{X \times Y}{=} (a, b(a))$$

where

$$\begin{aligned} b(x) &\stackrel{Y}{=} \delta(\lambda y.q(x, y)) \\ a &\stackrel{X}{=} \varepsilon(\lambda x.q(x, b(x))) \end{aligned}$$

Theorem on Finite Product of Selection Functions

Given sequence $\varepsilon \colon \Pi_{i \leq n} J_R X_i$, define

$$\left(\bigotimes_{i=0}^{n} \varepsilon_i \right) = \varepsilon_0 \otimes \ldots \otimes \varepsilon_n \quad : J_R \Pi_{i \leq n} X_i$$

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Theorem

Let $s = \left(\bigotimes_{i=0}^{n} \varepsilon_i\right)(q)$ *with* $q\colon \Pi_{i=0}^{n} X_i \to R$. *For* $0 \leq i \leq n$

$$s_i \stackrel{X_i}{=} \varepsilon_i p_i$$

$$qs \stackrel{R}{=} p_i(\varepsilon_i p_i)$$

for some $p_i\colon X_i \to R$

Classical Arithmetic

We can prove (classically)

$$(+) \quad \forall i \leq n \exists x \forall y (Q_i(y) \to Q_i(x))$$

By finite choice (i.e. induction) we obtain

$$\exists s \forall i \leq n \forall y (Q_i(y) \to Q_i(s_i))$$

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Claim: Can simply take $s = \left(\bigotimes_{i=0}^{n} \varepsilon_i\right)(q)$

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ε_i witnessing interp. of $(+)$

Claim: Can simply take $s = (\bigotimes_{i=0}^{n} \varepsilon_i)(q)$

product of sel. fcts.

Proof of Claim

From theorem on product of selection functions we have:

$$s_i \stackrel{X_i}{=} \varepsilon_i p_i$$
$$qs \stackrel{R}{=} p_i(\varepsilon_i p_i)$$

In order to produce s such that

$$\forall i \leq n(Q_i(qs) \to Q_i(s_i))$$

it is enough to find ε_i such that for all p

$$\forall i \leq n(Q_i(p(\varepsilon_i p)) \to Q_i(\varepsilon_i p))$$

(which is easy, as we have seen!)

Classical Analysis

What about infinitely many “uses” of classical logic?

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by **countable choice** we have

$$\exists \alpha \forall n \forall y (Q_n(y) \to Q_n(\alpha(n)))$$

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Given

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by **countable choice** we have

$$\exists \alpha \forall n \forall y (Q_n(y) \to Q_n(\alpha(n)))$$

whose *dialectica* interpretation (of negative translation) is

$$\forall \psi \forall q \exists \alpha \forall n \leq \psi \alpha (Q_n(q\alpha) \to Q_n(\alpha(n)))$$

where $\psi \colon X^\omega \to \mathbb{N}$ and $q \colon X^\omega \to X$

Controlled Iterated Product

This can be solved by a "controlled" iterated product

$$\left(\bigotimes_{s}^{\psi} \varepsilon\right)(q) \stackrel{R}{=} \begin{cases} \mathbf{0} & \psi(\hat{s}) < |s| \\ \left(\varepsilon_{|s|} \otimes \lambda x^{X_{|s|}}.\left(\bigotimes_{s*x}^{\psi} \varepsilon\right)\right)(q) & \text{otherwise} \end{cases}$$

Theorem

Let $\alpha \stackrel{X^\omega}{=} \left(\bigotimes_{\langle\rangle}^{\psi} \varepsilon\right)(q)$. *There exist* $p_i \colon X_i \to R$ *s.t.*

$$\begin{aligned} \alpha(i) &\stackrel{X_i}{=} \varepsilon_i(p_i) \\ q\alpha &\stackrel{R}{=} p_i(\varepsilon_i p_i) \end{aligned}$$

for all $i \leq \psi(\alpha)$

Outline

Back to $(\mathbb{N}^{\mathbb{N}} \to \mathbb{N})$-Example (I)

Corollary

For any $H\colon (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ *there exist* $f, g\colon \mathbb{N} \to \mathbb{N}$ *such that*

$$f \neq g \quad \textit{and} \quad H(f) \stackrel{\mathbb{N}}{=} H(g)$$

Proof.

Let $\alpha^{\mathbb{N} \to \mathbb{N}^{\mathbb{N}}}$ be some inverse of H, i.e. for all f and k

$$(*) \quad H(\alpha(k)) = k \qquad \text{if} \;\; H(f) = k$$

(using **classical logic** and **countable choice**)

Let $f_\alpha = \lambda n.\alpha(n)(n) + 1$ and $g_\alpha = \alpha(k_\alpha)$ where $k_\alpha = H(f_\alpha)$

Clearly $f_\alpha(k_\alpha) \neq g_\alpha(k_\alpha)$ and $H(f_\alpha) = k_\alpha \stackrel{(*)}{=} H(g_\alpha)$ □

Back to $(\mathbb{N}^{\mathbb{N}} \to \mathbb{N})$-Example (I)

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Back to $(\mathbb{N}^{\mathbb{N}} \to \mathbb{N})$-Example (I)

Construct approximation to inverse of H, i.e. $\alpha^{\mathbb{N} \to \mathbb{N}^{\mathbb{N}}}$ s.t.

$$\forall k \leq H(f_\alpha) \left(\underbrace{H(f_\alpha) = k \ \to \ H(\alpha(k)) = k}_{A_k(\alpha(k), f_\alpha)} \right)$$

Back to $(\mathbb{N}^{\mathbb{N}} \to \mathbb{N})$-Example (I)

Construct approximation to inverse of H, i.e. $\alpha^{\mathbb{N} \to \mathbb{N}^{\mathbb{N}}}$ s.t.

$$\forall k \leq H(f_\alpha) \left(\underbrace{H(f_\alpha) = k \ \to \ H(\alpha(k)) = k}_{A_k(\alpha(k), f_\alpha)} \right)$$

Enough to produce ε_k such that for all p

$$\underbrace{H(p(\varepsilon_k p)) = k \ \to \ H(\varepsilon_k p) = k}_{A_k(\varepsilon_k p, p(\varepsilon_k p))}$$

We have just built such ε_k's!

Back to $(\mathbb{N}^{\mathbb{N}} \to \mathbb{N})$-Example (II)

Let ε_i as before and $f_\alpha := \lambda n.\alpha(n)(n) + 1$

Theorem

Fix $H : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$. Let $q\alpha = f_\alpha$ and $\psi\alpha = H(f_\alpha)$. Define

$$\alpha = \left(\bigotimes_{\langle\,\rangle}^{\psi} \varepsilon \right) (q)$$

and $f = f_\alpha$ and $g = \alpha(\psi\alpha)$. Then

$$Hf = Hg \qquad \text{and} \qquad f(\psi\alpha) \neq g(\psi\alpha)$$

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