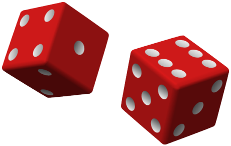
Proofs and Games



Paulo Oliva

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Classical Logic and Computation

Warwick, 8 July 2012

GAMES	LOGIC

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Game	Formula

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Game	**Formula**
Players	Proponent/Opponent

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Rules + Adjudication	Formal system

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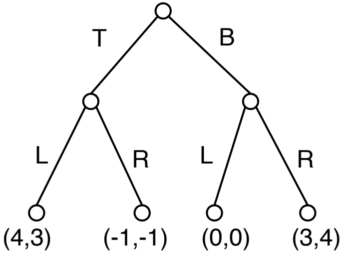
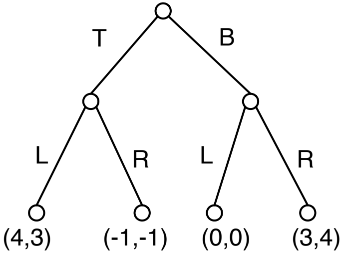
Winning Strategy	**Proof**

Extensive Form versus Strategic Form



Extensive form

Extensive Form versus Strategic Form



Extensive form

	LL	LR	RL	RR
T	(4,3)	(4,3)	(-1,-1)	(-1,-1)
B	(0,0)	(3,4)	(0,0)	(3,4)

Strategic form

Outline

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Lorenzen Games

- Lorenzen (1961)

- Two players $\{\mathbf{P}, \mathbf{O}\}$ debating about the truth of a formula
- Players take turns attacking or responding
- A player wins if the other can't attack or respond

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- Players take turns attacking or responding
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- Motivation: alternative semantics for IL
  If formula is provable in IL then **P** has winning strategy
- Felscher (1985) found conditions for completeness
  Formula is provable in IL iff **P** has winning strategy

Lorenzen Games – E.g. $P \wedge Q \to Q \wedge P$

Possible play in this game:

(0) **P** starts by asserting $P \wedge Q \to Q \wedge P$

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(4)	**P** attacks (1) asserting	$\wedge_2$

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Possible play in this game:

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(4)	**P** attacks (1) asserting	$\wedge_2$
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Lorenzen Games – Structural Rules

General organisation of the game:

S1 **P** may only assert atomic formulas already asserted by **O**

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General organisation of the game:

- S1 **P** may only assert atomic formulas already asserted by **O**
- S2 A player can only respond the latest open attack
- S3 An attack may be responded at most once
- S4 A **P**-assertion may be attacked at most once

Outline

Blass Games

Blass'1992

Games for **affine logic** (linear logic plus weakening)

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- Two kinds of connectives (only one re-attackable)

Can dispense with structural rule!

Blass Games – Definition

Two players **P** and **O**

A **Blass game** is a triple $\mathcal{G} = (M, p, G)$ where

- M is the set of **possible moves** at each round
- $p \in \{$**P**, **O**$\}$ is the **starting player**
  (from then on players take turns)
- $G\colon M^\omega \to \{$**P**, **O**$\}$ is the **outcome function**

Blass Games – Conjunctions

Given games $\mathcal{G}_0 = (M_0, s_0, G_0)$ and $\mathcal{G}_1 = (M_1, s_1, G_1)$

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- **O** starts and chooses $i \in \{0, 1\}$
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The game $\mathcal{G}_0 \otimes \mathcal{G}_1$. Defined as

- play both games **interleaved**
- **O**'s turn in $\mathcal{G}_0 \otimes \mathcal{G}_1$ if it's his turn in **both** $\mathcal{G}_0$ and $\mathcal{G}_1$
  He chooses one of the games and makes a move there
- **P**'s turn in $\mathcal{G}_0 \otimes \mathcal{G}_1$ if his turn in one of $\mathcal{G}_0$ or $\mathcal{G}_1$
  He must play on the sub-game where it's his turn
- **O** wins iff he wins in at least one of $\mathcal{G}_0$ or $\mathcal{G}_1$

Blass Games

- The dual of a game is simply a swapping of roles

$$\mathcal{G}^{\perp} = (M, \overline{s}, \overline{G})$$

- Given game interpretation of atomics $P \mapsto \mathcal{G}_P$ extend to game interpretation $\mathcal{G}_A$ for all formulas A

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Theorem (Blass,1992)

A is provable in affine logic $\Rightarrow$ **P** *has winning strategy in $\mathcal{G}_A$*
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- **Abramsky and Jagadeesan'1992**
  Soundness and completeness for MLL + mix rule
- **Hyland and Ong'1993**
  Soundness and completeness for MLL

Outline

It is my thesis that game-theoretically inspired conceptualizations have much to offer in other parts of logical studies as well. An especially neat case in point is offered by Gödel's functional interpretation of first-order arithmetic. As Dana Scott first pointed out, by far the most natural way of looking at it is in game-theoretical terms.

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Our category of games is a special case of a general construction in the appendix to Barr's book [1]. It is closely related to de Paiva's *dialectica categories* [10,11].

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In developing a category-theoretic approach to the Dialectica interpretation, de Paiva [3] found a connection with linear logic. This connection suggests looking at the Dialectica interpretation, in de Paiva's category-theoretic version, from the point of view of game semantics, and this is the purpose of the present section.

Blass, *A game semantics for LL*, 1992

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Repeated applications turns long games

$$\forall x_0 \exists y_0 \ldots \forall x_n \exists y_n Q(x_0, y_0, \ldots, x_n, y_n)$$

into **two-round games**

$$\exists f_0 \ldots f_n \forall x_0 \ldots x_n Q(x_0, f_0(x_0), \ldots, x_n, f_n(\vec{x}))$$

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P chooses $t = \langle t_0 \ldots t_n \rangle$, then **O** chooses $s = \langle s_0 \ldots s_n \rangle$

P wins iff $Q(s_0, t_0(s_0), \ldots, s_n, t_n(\vec{s}))$

Finite Types and System T

Finite types generated by

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Gödel primitive recursor

$$\mathsf{R}(x, f, n) \stackrel{X}{=} \begin{cases} x & \text{if } n = 0 \\ f(n-1, \mathsf{R}(x, f, n-1)) & \text{if } n > 0 \end{cases}$$

where X is an any finite type

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Gödel's system T: Primitive recursive functionals

Remark: Ackermann function definable using $X = \mathbb{N}^{\mathbb{N}}$

Strategic-form Games

Identify $\mathbb{B} = \{\mathbf{P}, \mathbf{O}\}$

Formula A assigned a **game** with **outcome function**

$$|A| : X \times Y \to \mathbb{B}$$

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Theorem (Gödel, 1958)

$$\mathsf{HA} \vdash A \quad \stackrel{\exists t \in \mathbf{T}}{\Longrightarrow} \quad \mathbf{T} \vdash \forall y |A|^t_y$$

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Assume $|A| : X \times Y \to \mathbb{B}$ and $|B| : V \times W \to \mathbb{B}$ defined.

Then:

$$|A \wedge B|^{\langle x,v \rangle}_{\langle y,w \rangle} \quad \equiv \quad |A|^x_y \wedge |B|^v_w$$

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|A \vee B|_{\langle y,w\rangle}^{\mathsf{inj}_b x} &\equiv \begin{cases} |A|_y^x & \text{if } b = l \\ |B|_w^x & \text{if } b = r \end{cases}
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|\forall z A|_{\langle a,y\rangle}^f & \equiv & |A[a/z]|_y^{fa} \\
|A \to B|_{\langle x,w\rangle}^{\langle f,g\rangle} & \equiv & |A|_{gxw}^x \to |B|_w^{fx}
\end{array}
$$

Functional interpretations

Strategic-form game above is **dialectica interpretation**

$$\boxed{|A|^x_y \;\equiv\; A_D(x;y)}$$

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Variant where interpretation of implication is changed to

$$|A \to B|^f_{\langle x,w\rangle} \;\equiv\; \forall y |A|^x_y \to |B|^{fx}_w$$

gives Kreisel's **modified realizability**

$$\boxed{\forall y |A|^x_y \;\equiv\; x \;\mathbf{mr}\; A}$$

Functional interpretations – Linear logic

P and **O** choose moves simultaneously!

Assume $|A| : X \times Y \to \mathbb{B}$ and $|B| : V \times W \to \mathbb{B}$ defined

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 & \text{or} & \forall y \in fx \, |A|^{x}_{y} & \text{(Diller-Nahm variant)}
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Outline

Extensive-form Game (Perfect info, No chance player)

An **extensive form game** consists of

- A set of n **players**
- A tree T, called the **game tree**
- A **payoff function** $q\colon T_{\text{leaf}} \to \mathbb{R}^n$
  (T_{leaf} = leaves of T)
- A **partition of the non-terminal nodes** into n subsets

Generalising "Goal"

Usually:

X = set of choices

$\mathbb{R}$ = payoffs

Maximise return

$$\max \in (X \to \mathbb{R}) \to \mathbb{R}$$

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More generally:

X = set of possible moves

R = set of outcomes

"**Quantifier**"

$$\phi \in \underbrace{(X \to R) \to 2^R}_{K_R X}$$

Generalising "Goal"

Usually:

X = set of choices

$\mathbb{R}$ = payoffs

Maximise return

$$\max \in (X \to \mathbb{R}) \to \mathbb{R}$$

More generally:

X = set of possible moves

R = set of outcomes

"**Quantifier**"

$$\phi \in \underbrace{(X \to R) \to 2^R}_{K_R X}$$

Other Quantifiers: $\exists, \forall, \sup, \inf, \min, \max, \int_0^1$, fix

Extensive-form Game (Generalised)

No players! (at least not explicitly)

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An **extensive form game** is described by

- A labelled tree T, called the game tree
  (X_s = labels on branching at position s)
- A set of **outcomes** R
- **Quantifiers** $\phi_s \colon K_R X_s$ for each position s
- An **outcome function** $q \colon T_{\text{leaf}} \to R$
  (T_{leaf} = leaves of T)

Definition (Strategy)

Choice of move for each position, i.e.

$$\mathsf{next} \colon \Pi_{s \in T} X_s$$

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Definition (Optimal Strategy)

A strategy is **optimal** if for any position s we have

$$q(s * \alpha^s) \in \phi_s(\lambda x. q(s * x * \alpha^{s*x}))$$

Quantifiers and Selection Functions

Functionals $\varepsilon \colon \underbrace{(X \to R) \to X}_{J_R X}$ are called **selection functions**

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J_R and K_R are **strong monads**, so we have $F \in \{J_R, K_R\}$

$$\otimes : \; FX \times (X \to FY) \to F(X \times Y)$$

product operations on selection functions and quantifiers

Iterated Products

Iterated product of quantifiers

$$\left(\bigotimes_s^T \phi\right)(q) \stackrel{R}{=} \begin{cases} q([\,]) & \text{if } T_{\text{leaf}}(s) \\ \left(\phi_s \otimes \lambda x.\left(\bigotimes_{s*x}^T \phi\right)\right)(q) & \text{otherwise} \end{cases}$$

where q is the outcome function of sub-game at position s

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Spector's BR $\equiv$ Restricted BR, over system T [O./Powell'12]

Sequential Games – Main Result

Fix an unbounded game $G = (T, R, \phi, q)$

Assume $\phi_s \colon K_R X_s$ attainable with selection fcts $\varepsilon_s \colon J_R X_s$

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Backward induction @ Game Theory $(\phi = \sup)$

Bekič's lemma @ Fixed Point Theory $(\phi = \mathrm{fix})$

Backtracking @ Algorithms $(\phi = \exists)$

Bar recursion @ Proof Theory

Countable Choice (dialectica interpretation)

Let us look at negative translation of countable choice:

$$\Pi_1\text{-}\mathsf{AC}_0^N \;:\; \forall n \neg\neg \exists x A_n(x) \to \neg\neg \exists \alpha \forall n A_n(\alpha n)$$

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determines tree **quantifier at round n**

Countable Choice (dialectica interpretation)

Interpretation of $\mathrm{AC}_0 \equiv$ Game in extensive form

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Given $|A_n(x)|_y$ and selection fcts. ε_n define quantifiers

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Corollary

Given $A_n(x)$, a witness α for dialectica interpretation of Π_1-AC_0^N can be calculated as

$$\alpha = \left(\bigotimes_s^T \varepsilon \right) (q')$$

where $T_{\text{leaf}}(s) \equiv \omega(s * \mathbf{0}) < |s|$ *and* $q'(s) = q(s * \mathbf{0})$

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