Games and Logic

Paulo Oliva

Queen Mary University of London

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Theorem

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Proof.

One of $n = 0$ and $n = f(0)$ and ... and $n = f^{f0}(0)$ works, as the following can't happen

$$f0 > f^2 0 > \ldots > f^{f0} 0$$

Games	**Logic**
Game	
Players	
Rules + Adjudication	
Play	
Strategy	
Winning Strategy	

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Outline

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Lorenzen Games

- Lorenzen (1961)
- Two players $\{\mathbf{P}, \mathbf{O}\}$ debating about the truth of a formula
- Players take turns attacking or responding
- A player wins if the other can't attack or respond

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- Two players $\{\mathbf{P}, \mathbf{O}\}$ debating about the truth of a formula
- Players take turns attacking or responding
- A player wins if the other can't attack or respond
- Motivation: alternative semantics for **intuitionistic logic**
  Formula is provable in IL iff **P** has winning strategy

Lorenzen Games – E.g. $P \wedge Q \to Q \wedge P$

Possible play in this game:

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R3 **P** may only assert atomic formulas already asserted by **O**

Lorenzen Games – Intuition

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P *chooses path from below, directed by* **O***-attacks*

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Can dispense with structural rule!

Blass Games – Definition

Two players **P** and **O**

A **Blass game** is a triple $\mathcal{G} = (M, p, G)$ where

- M is the set of possible moves at each round
- $p \in \{$**P**, **O**$\}$ is the starting player
- $G\colon M^\omega \to \mathbb{B}$ is the outcome function
  $G(\alpha) =$ true *means* **P** *wins**

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The new game $\mathcal{G}_0 \otimes \mathcal{G}_1$ is defined as

- both games are played intertwined
- **O** plays when its his turn in both sub-games
  He chooses one of the games and makes a move there
- **P** plays when he is to move in either $\mathcal{G}_0$ or $\mathcal{G}_1$
- **O** wins if he wins in one of the sub-games

Blass Games

- The dual of a game is simply a swapping of roles $\mathcal{G}^\perp = (M, \overline{p}, \overline{G})$
- Given game interpretation of atomics $P \mapsto \mathcal{G}_P$ extend to game interpretation $\mathcal{G}_A$ for all formulas

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Theorem (Blass,1992)

*A is provable in affine logic $\Rightarrow$ **P** has winning strategy in $\mathcal{G}_A$*
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- **Abramsky and Jagadeesan'1992**
  Soundness and completeness for MLL + mix rule
- **Hyland and Ong'1993**
  Soundness and completeness for MLL

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What if we could allow for higher-order moves?

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Repeated applications turns long games

$$\forall x_0 \exists y_0 \ldots \forall x_n \exists y_n Q(x_0, y_0, \ldots, x_n, y_n)$$

into **two-round games**

$$\exists f_0 \ldots f_n \forall x_0 \ldots x_n Q(x_0, f_0(x_0), \ldots, x_n, f_n(\vec{x}))$$

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$$\exists f_0 \ldots f_n \forall x_0 \ldots x_n Q(x_0, f_0(x_0), \ldots, x_n, f_n(\vec{x}))$$

P chooses $t = \langle t_0 \ldots t_n \rangle$, then **O** chooses $s = \langle s_0 \ldots s_n \rangle$

P wins iff $Q(s_0, t_0(s_0), \ldots, s_n, t_n(\vec{s}))$

Finite types:

$$X, Y \;:\equiv\; \mathbb{B} \mid \mathbb{N} \mid X \times Y \mid X \uplus Y \mid Y^X$$

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Theorem (Gödel, 1958)

$$\mathsf{HA} \vdash A \quad \stackrel{\exists t \in \mathbf{T}}{\Longrightarrow} \quad \mathbf{T} \vdash \forall y |A|^t_y$$

Higher-order Games

Let $|A|: X \times Y \to \mathbb{B}$ and $|B|: V \times W \to \mathbb{B}$ given. Then:

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Winning strategy $\Rightarrow$ strategy profile in equilibrium

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For instance:

X = savings accounts

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Other examples: $\exists, \forall, \sup, \int_0^1$, fix, $\ldots$

Quantifiers and Selection Functions

Functionals $\varepsilon \colon \underbrace{(X \to R) \to X}_{J_R X}$ are called **selection functions**

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K and J are **strong monads**, so we have $T \in \{K_R, J_R\}$

$$TX \times TY \to T(X \times Y)$$

a **product operation** on selection functions and quantifiers

Sequential Games

A **sequential game with n rounds** is described by

- Sets of **available moves** X_i for each round $1 \leq i \leq n$
- A set of **outcomes** R
- **Quantifiers** $\phi_i \colon K_R X_i$ for each round $1 \leq i \leq n$
- An **outcome function** $q \colon \Pi_{i=1}^{n} X_i \to R$

Recent Work

(joint with Martín Escardó and Thomas Powell)

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- Unbounded product equivalent to Spector's bar recursion
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- View theorems as generalised von Neumann games
  View proofs as calculations of opt. strat. in such games