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Darmstadt talk 06/12/11

System T types $X, Y := \mathbb{N} \mid \mathbb{B} \mid X \to Y \mid X \times Y \mid X + Y \mid X^* \mid$

Quantifiers: types of the kind $(X \to R) \to R \equiv KX$

E.g. $\exists : (X \to \mathbb{B}) \to \mathbb{B}$

$\inf : (X \to \mathbb{N}) \to \mathbb{N}$

Selection functions: $(X \to R) \to X \equiv JX$

E.g. $\varepsilon : (X \to \mathbb{B}) \to X$

$\operatorname{argmin} : (X \to \mathbb{N}) \to X$

interpretation of $\neg\neg A$

$\neg\neg \exists x \forall r\, A_D(x,r) \mapsto \exists \varepsilon \forall p\, A_D(\varepsilon p, p\varphi)$

$\neg\neg \exists x\, A_{mr}(x) \mapsto \exists \phi \forall p\, \forall x (A_{mr}(x) \to R(px)) \to R\, \phi p$

Attainable quantifiers: $\phi : KX$ attainable if $\exists \varepsilon : JX$ s.t.

$$\forall p\, (\phi p = p(\varepsilon p))$$

Embedding $\overline{(\cdot)} : JX \to KX$

$$\bar{\varepsilon} p = p(\varepsilon p)$$

Product of quantifiers $\otimes : KX \times (X \to KY) \to K(X \times Y)$

$$(\phi \otimes \psi)(q) = \phi(\lambda x.\, \psi_x(\lambda y.\, qxy))$$

E.g. $(\exists \otimes \forall)(q) = \exists x \forall y\, qxy$

Product of selection functions $\otimes : JX \times (X \to JY) \to J(X \times Y)$

$$(\varepsilon \otimes \delta)(q) = (a, B(a))$$

$$a = \varepsilon(\lambda x.\, q(x, Bx))$$

$$B(x) = \delta_x(\lambda y.\, qxy)$$

Lemma: $\overline{\varepsilon \otimes \delta} = \bar{\varepsilon} \otimes \lambda x.\, \bar{\delta}$

Proof: $(\bar{\varepsilon} \otimes \lambda x.\, \bar{\delta})(q) = \bar{\varepsilon}(\lambda x.\, \bar{\delta}_x(\lambda y.\, qxy))$

$= \bar{\varepsilon}(\lambda x.\, q(x, Bx))$

$= q(a, B(a))$

$= q((\varepsilon \otimes \delta)(q))$

$= \overline{(\varepsilon \otimes \delta)}(q)$ $\square$

Unbounded product of quantifiers

Let $\phi = (\phi_s)_{s:X^*}$ and $\omega : X^{\mathbb{N}} \to \mathbb{N}$ and $q : X^{\mathbb{N}} \to R$

$$EPQ_s^{X,R}(\omega)(\phi)(q) \overset{R}{=} \begin{cases} 0 & \text{if } \omega(\hat{s}) < |s| \\ \underbrace{(\phi_s \otimes \lambda x.\, EPQ_{s*x})(q)}_{\phi_s(\lambda x.\, EPQ_{s*x}(q_x))} & \text{else} \end{cases}$$

<u>Lemma</u>: Given $\omega : X^{\mathbb{N}} \to \mathbb{N}$ and $\alpha : X^{\mathbb{N}}$ let $n = EPQ_{\langle\rangle}(\omega)(\phi)(0)$

where $\phi_s(p) = p(\alpha(|s|)) + 1$

then $\exists i \leq n.\ \omega(\overline{\alpha i}) < i$

<u>Proof</u>: note that
$$EPQ_{\overline{\alpha n}} = \begin{cases} 0 & \text{if } \omega(\overline{\alpha n}) < n \\ 1 + EPQ_{\overline{\alpha(n+1)}} & \text{else.} \end{cases}$$

Unbounded product of selection functions

Let $\varepsilon = (\varepsilon_s)_{s:X^*}$ and $\omega : X^{\mathbb{N}} \to \mathbb{N}$ ~~and $q : X^{\mathbb{N}} \to R$~~

$$EPS_s^{X,R}(\omega)(\varepsilon) \overset{TX^{\mathbb{N}}}{=} \begin{cases} 0 & \text{if } \omega(\hat{s}) < |s| \\ \varepsilon_s \otimes \lambda x.\, EPS_{s*x} & \text{else} \end{cases}$$

$$EPS_s(\omega)(\varepsilon)(q) \overset{X^{\mathbb{N}}}{=} \begin{cases} 0 & \text{if } \omega(\hat{s}) < |s| \\ a * \underbrace{B(a)}_{EPS_{s*a}(q_a)} \end{cases}$$

$$a = \varepsilon_s(\lambda x.\, q(x, Bx)) = \varepsilon_s(\underbrace{\lambda x.\, q(x * EPS_{s*x}(q_x))}_{\lambda x.\, \overline{EPS}_{s*x}(q_x)})$$

$$B(x) = EPS_{s*x}(q_x)$$

<u>Lemma</u> EPS with $q = id$ is T-equiv. to EPS

<u>Proof</u>: $\tilde{\varepsilon} p = p(\lambda x.\, q(px))$ $\begin{cases} \varepsilon : (X \to R) \to X & q : X^{\mathbb{N}} \to R \\ \tilde{\varepsilon} : (X \to X^{\mathbb{N}}) \to X & \underset{id}{q} : X^{\mathbb{N}} \to X^{\mathbb{N}} \end{cases}$

<u>Lemma</u>: $\overline{EPS_s(\omega)(\varepsilon)} \overset{KX^{\mathbb{N}}}{=} EPQ_s(\omega)(\tilde{\varepsilon})$

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<u>Corollary</u> EPQ T-defines EPS

<u>Proof</u>: Given ε, ω define $(R = X^{\mathbb{N}})$

$$EPS_s^X(\omega)(\varepsilon) \overset{X^{\mathbb{N}}}{=} EPQ_s^{X, X^{\mathbb{N}}}(\omega)(\bar{\varepsilon})(id)$$

<u>Question</u>: Is EPQ T-definable in EPS? YES!

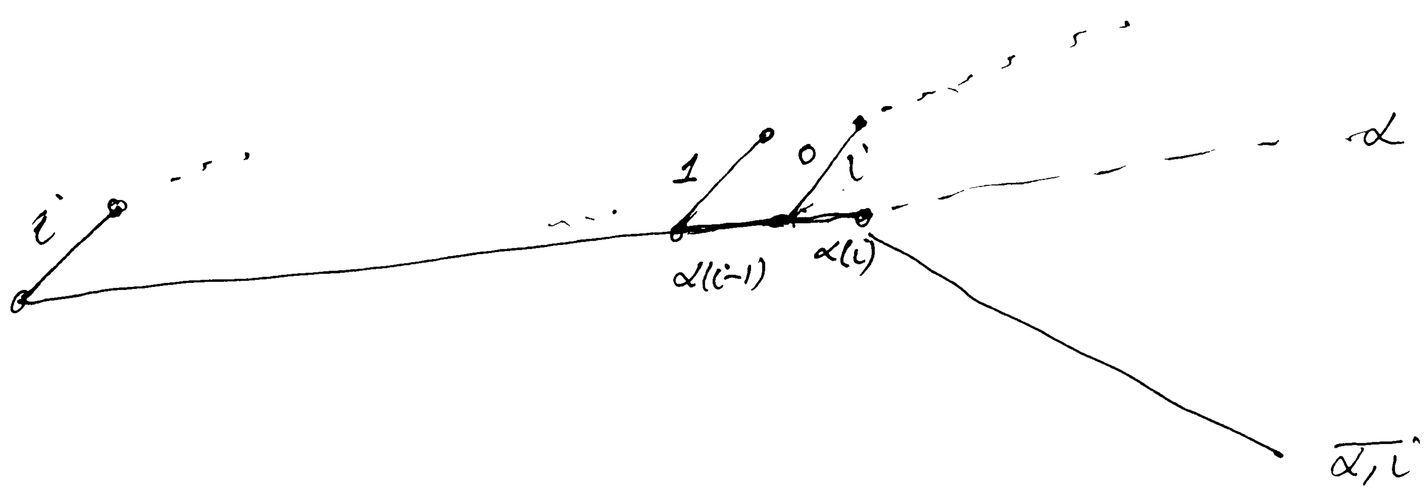
<u>Lemma</u> Given $\omega : X^{\mathbb{N}} \to \mathbb{N}$ and $\alpha : X^{\mathbb{N}}$ let

$$n = EPS_{\langle\rangle}(\omega)(\varepsilon)(0^{\mathbb{N}})(0^X) \quad \text{with } \varepsilon_s p \overset{X}{=} 0$$

then where

$$\varepsilon_s p \overset{X}{=} \begin{cases} 0 & \text{if } \exists j \leq |s| \ (\omega(\alpha, j) < j) \\ 1 + p(\alpha(|s|))(|s|+1) & \text{else} \end{cases}$$

<u>Proof</u>:



<u>Theorem</u>: EPQ is T-definable in EPS

<u>Proof</u>: define $EPQ^{X,R}$ using EPS^{X+R}

$$\phi_s : KX \Rightarrow \varepsilon_s : J(X+R)$$

$$\varepsilon_s p \overset{X+R}{=} \left[\phi_s(\lambda x . p([x]_X))\right]_R$$

$$\alpha : (X+R)^{\mathbb{N}} \Rightarrow \tilde{\alpha} : X^{\mathbb{N}}$$

$$q : X^{\mathbb{N}} \to R \Rightarrow \tilde{q} : (X+R)^{\mathbb{N}} \to R$$

$$\tilde{q}(\alpha) \overset{R}{=} \begin{cases} q\tilde{\alpha} & \text{if } \forall i < N \ \alpha(i) \in X \\ \alpha(i) & \text{if } \exists i < N \ \alpha(i) \in R \end{cases}$$

N = least i s.t. $\omega(\alpha, i) < i$

$$EPQ_s(\omega)(\phi)(q) = \overline{EPS_s(\tilde{\omega})(\bar{\varepsilon})(\tilde{q})}$$

$\square$