Nash Equilibrium
Bekič's Lemma and
Bar Recursion

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(based on jww Martín Escardó and Thomas Powell)

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Germany, 11 November 2011

Outline

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Sequential Payoff Games

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- **strategy** for player i is a mapping

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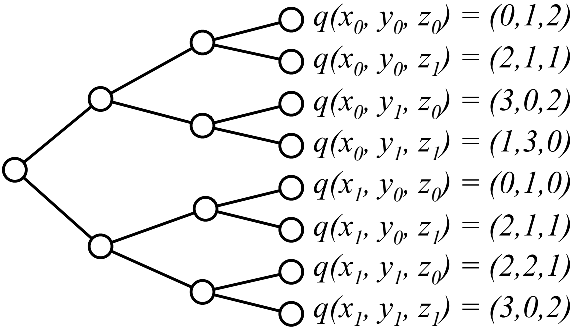
- **strategy profile** is a tuple $(\text{next}_i)_{1 \le i \le n}$
- A strategy profile is in (Nash) **equilibrium** if no single player has an incentive to unilaterally change his strategy

Backward Induction

Three players, payoff function $q\colon X \times Y \times Z \to \mathbb{R}^3$

Each player is trying to maximise their own payoff



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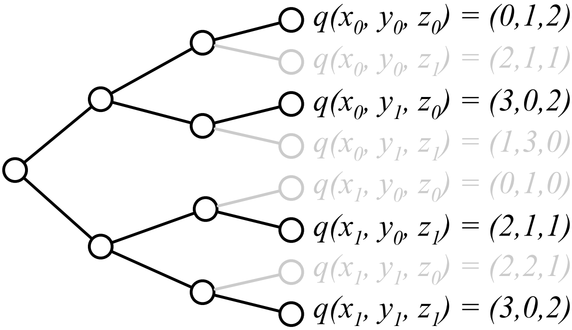

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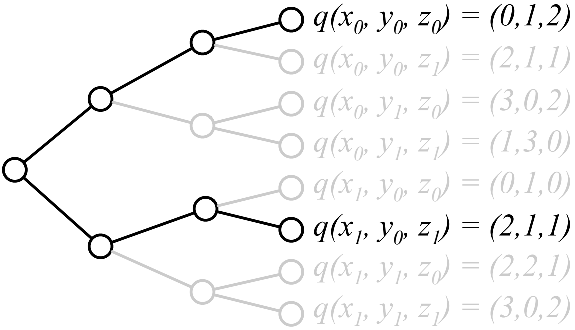

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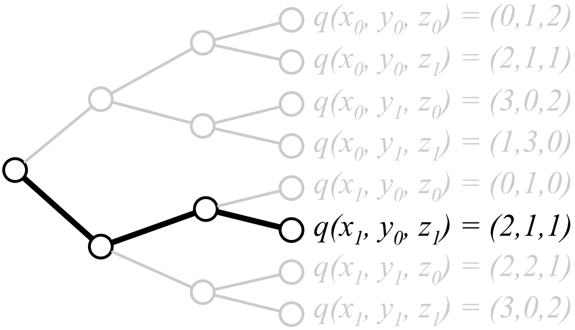

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BI: $\Pi_{j \leq i} X_j \to \Pi_{j > i} X_j$

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find $x \in X_i$ where $p \colon X_i \to \mathbb{R}^n$ has maximal i-value

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fix payoff function $q \colon \Pi_{i=1}^n X_i \to \mathbb{R}^n$

$$\mathsf{BI}(s) \overset{\Pi_{j>|s|} X_j}{=} \begin{cases} [\,] & \text{if } n = |s| \\ c_s * \mathsf{BI}(s * c_s) & \text{otherwise} \end{cases}$$

where $c_s = \mathrm{argmax}_{|s|+1}(\lambda x. q(s * x * \mathsf{BI}(s * x)))$

Equilibrium Strategy Profile

Let

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where $c_s = \mathrm{argmax}_{|s|+1}(\lambda x.q(s * x * \mathsf{BI}(s * x)))$

Each player's **optimal strategy** can be described as

$$\mathsf{next}_i(s) = \mathrm{argmax}_i(\underbrace{\lambda x.q(s * x * \mathsf{BI}(s * x))}_{p\colon X_i \to \mathbb{R}^n})$$

Outline

Bekič's Lemma

Theorem

If each space X_i *has a* **fixed point operator**

$$\mathsf{fix}_i : (X_i \to X_i) \to X_i$$

then so does the product space $X_1 \times \ldots \times X_n$

Bekič's Lemma – Construction

$\mathsf{BL}\colon \Pi_{j\leq i} X_j \to \Pi_{j>i} X_j$

fixed point over $\Pi_{j>i} X_j$ assuming $s\colon \Pi_{j\leq i} X_j$ fixed

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$\widetilde{\mathsf{fix}}_i\colon (X_i \to \Pi_{j=1}^{n}X_j) \to X_i$

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given $q\colon \Pi_{i=1}^n X_i \to \Pi_{i=1}^n X_i$

$$\mathsf{BL}(s) \overset{\Pi_{j>|s|} X_j}{=} \begin{cases} [\,] & \text{if } n = |s| \\ c_s * \mathsf{BL}(s * c_s) & \text{otherwise} \end{cases}$$

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Hence, a fixed point of q is

$$\mathsf{BL}([\,]) = [x_1, \ldots, x_n]$$

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Interpreting Finite Choice

Finite Choice

$$\forall i \leq n \exists x \forall r A_i(x, r) \rightarrow \exists s \forall i \leq n \forall r A_i(s_i, r)$$

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$$\exists \varepsilon \forall i \leq n \forall p A_i(\varepsilon_i p, p(\varepsilon_i p)) \rightarrow \forall q \exists s \forall i \leq n A_i(s_i, qs)$$

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Problem

Given $\varepsilon_i \colon (X \rightarrow R) \rightarrow X$ *such that*

$$\forall i \le n \forall p A_i(\varepsilon_i p, p(\varepsilon_i p))$$

and $q \colon X^n \rightarrow R$ *produce* $s \colon X^n$ *such that*

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given "counter-example function" $q\colon X^* \to R$

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Take

$$s = \mathsf{BR}([\,])$$

Spector's Bar Recursion

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This is actually the **iterated product of selection functions**

T-equivalent to Spector's restricted form of bar recursion

Product of Selection Functions BR

EPS gives **direct** realisers as

- $\lambda\varepsilon, q, n.\mathsf{EPS}_{[\,]}(n)(\varepsilon)(q)$ realises

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  DC : $\forall s \exists x A_s(x) \to \exists \alpha \forall n A_{\overline{\alpha}n}(\alpha(n))$

Spector'62 first defines **general bar recursion**:

6.2. *Bar recursion.* For ease in reading we omit showing G, H, Y as arguments of ϕ.

$$\phi(x, C) = \begin{cases} G(x, \langle C0, \cdots, C(x-1)\rangle) \text{ if } Y(\langle\langle C0, \cdots, C(x-1)\rangle\rangle) < x\,, \\ H[\lambda a \phi(x', \langle C0, \cdots, C(x-1), a\rangle), x, \langle C0, \cdots, C(x-1)\rangle] \text{ otherwise.} \end{cases}$$

Thus $\phi(x, C)$ is defined outright if $Y(\langle\langle C0, \cdots, C(x-1)\rangle\rangle) < x$, and in terms of $\lambda a \phi(x', \langle C0, \cdots, C(x-1), a\rangle)$ otherwise.

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But only uses **restricted bar recursion**:

10. The interpretation of F is provable in Σ_4. This is the only point where we make use of bar recursion, and we use it in the following restricted form, where the parameter G_0 is not exhibited as an argument of ϕ for greater readability.

$$\text{BR} \qquad \phi z C x = \begin{cases} Cx & \text{if } x < z\,, \\ \mathbf{0} & \text{if } x \geqq z \wedge Y(\langle C0, \cdots, C(z-1)\rangle) < z\,, \\ \phi(z', \langle C0, \cdots, C(z-1), a_0 \rangle, x) & \text{otherwise}\,, \end{cases}$$

where

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Theorem (O./Thomas Powell)

The restricted and the general forms of Spector bar recursion are T-equivalent

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Theorem (O./Thomas Powell)

Product of quantifiers and product of selection functions are T-equivalent

Further Information

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On Spector's bar recursion
Final draft available