A Finitisation of the Infinite Ramsey Theorem

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(*talk based on joint work with M. Escardó and T. Powell*)

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Outline

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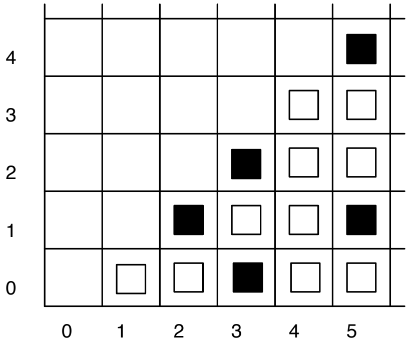
Given a colouring of sets $\{x, y\} \in \mathcal{P}_2(\mathbb{N})$ (with $x \neq y$)



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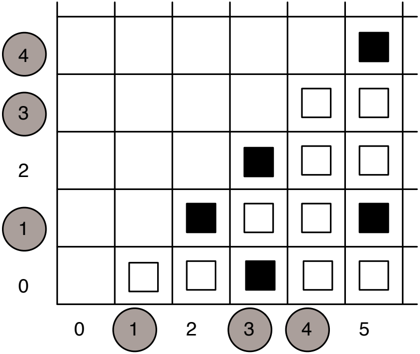

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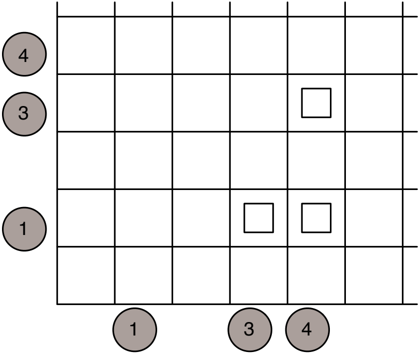

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$$\exists x^{\mathbb{B}} \underbrace{\exists F^{\mathbb{N}^{\mathbb{N}}}}_{\text{set}} \forall i \forall j < i \left(\underbrace{Fj < Fi}_{\text{infinite}} \wedge \underbrace{c(Fj, Fi) = x}_{\text{monochromatic}} \right)$$

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We shall witness equivalent "no-counterexample" variant

$$\forall \varepsilon \exists x^{\mathbb{B}} \underbrace{\exists F^{\mathbb{N}^{\mathbb{N}}}}_{\text{approx}} \forall i \leq \varepsilon_x F \, \forall j < i \left(\underbrace{Fj < Fi}_{\text{large}} \wedge \underbrace{c(Fj, Fi) = x}_{\text{monochromatic}} \right)$$

where $\varepsilon_x : (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$

The Erdős/Radu Tree

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$0 \prec 1$ and $j \prec i$ if $\forall k \prec j(c(k,j) = c(k,i))$

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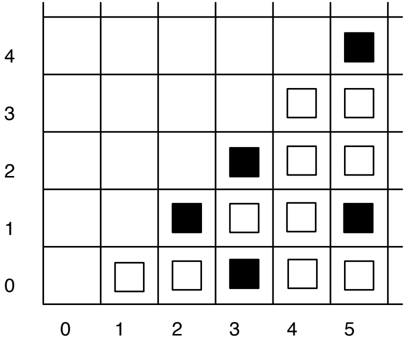

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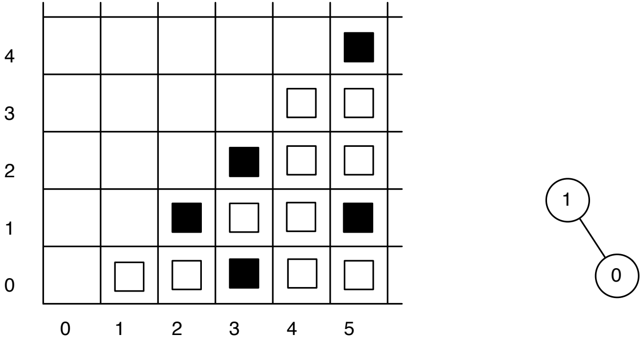

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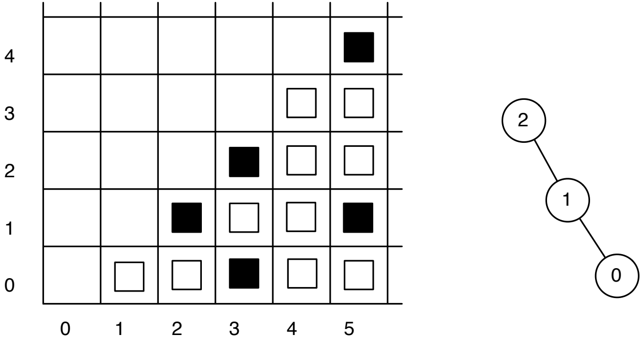

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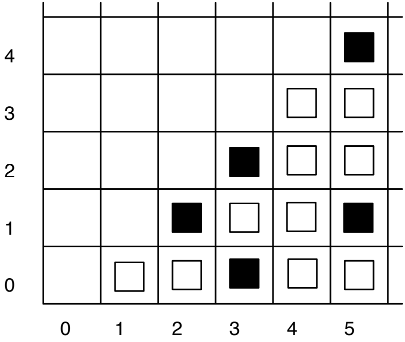

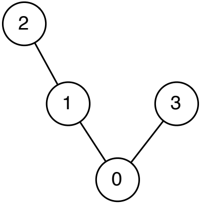

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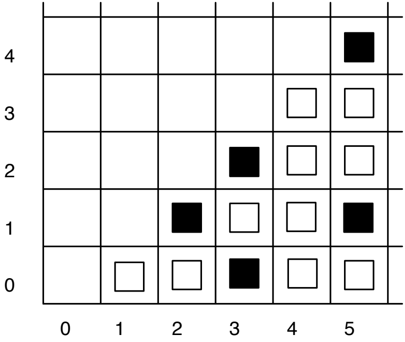

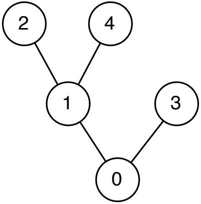

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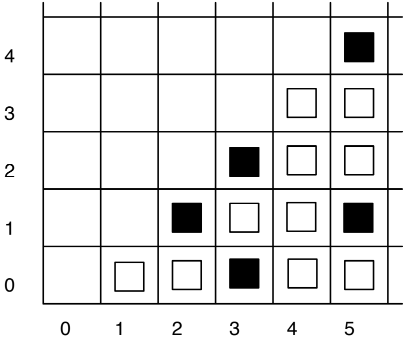

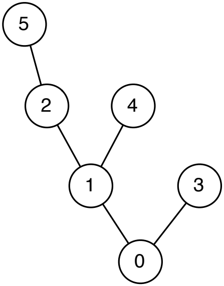

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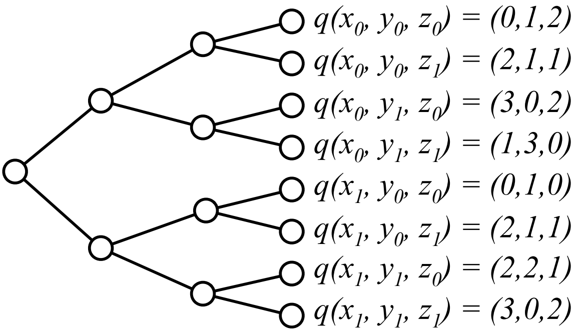
5. Hence, $\alpha \circ p$ is a monochromatic set for c

Outline

Nash Equilibrium – Backward Induction

Three players, payoff function $q\colon X \times Y \times Z \to \mathbb{R}^3$

Each player is trying to maximise their own payoff



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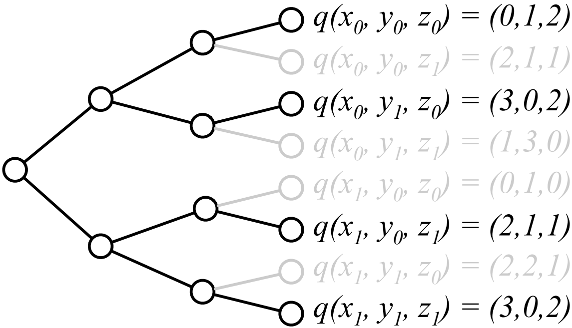

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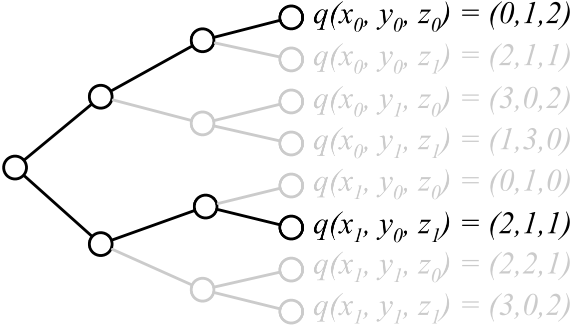

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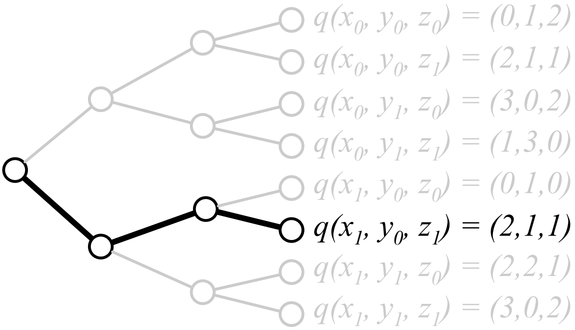

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$$\mathsf{BI}_i^{n-1}(q) \overset{\Pi_{j=i}^{n-1} X_j}{=} \begin{cases} [\,] & \text{if } i = n \\ c_i * \mathsf{BI}_{i+1}^{n-1}(q_{c_i}) & \text{otherwise} \end{cases}$$

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Each player's **optimal strategy** can be described as

$$\mathsf{next}_i(s) = \text{argmax}_i(\underbrace{\lambda x. q_{s*x}(\mathsf{BI}_{i+1}^n(q_{s*x}))}_{p \colon X \to \mathbb{R}^n})$$

Spector's Bar Recursion

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Using product $\otimes$ of **selection functions**

$$\mathsf{EPS}^\omega_s(\varepsilon) \overset{J_R X^*}{=} \begin{cases} \lambda q.[\,] & \text{if } \omega(\hat{s}) < |s| \\ \varepsilon_s \otimes \lambda x.\mathsf{EPS}^\omega_{s*x}(\varepsilon) & \text{otherwise} \end{cases}$$

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We have that

1. *there exist* $p_i \colon X \to R$, *for* $i < |s|$, *such that*

$$\begin{aligned} s_i &\stackrel{X}{=} \varepsilon_i p_i \\ qs &\stackrel{R}{=} p_i(\varepsilon_i p_i) \end{aligned}$$

2. $\omega \hat{s} < |s|$

Computational Interpretation

Main theorem gives straightforward interpretations:

Σ^0_1-comprehension	$\mapsto$	EPS for $\varepsilon \colon J_{\mathbb{N}}\mathbb{N}$
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Three nested calculations of **optimal strategies**

Outline

Infinite Pigeonhole Principle IPP

Let $\mathbf{n} = \{0, 1, \ldots, n-1\}$

Given a colouring $c \colon \mathbb{N} \to \mathbf{n}$ the principle **IPP** says

$$\exists k < n \quad \underbrace{\exists p^{\mathbb{N} \to \mathbb{N}}}_{\text{subsequence}} \quad \forall i(\quad \underbrace{pi \geq i}_{\text{unbounded}} \quad \wedge \quad \underbrace{c(pi) = k}_{\text{homogeneous}} \quad)$$

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We look at its **no-counterexample interpretation**

$$\forall \varepsilon \exists k < n \exists p^{\mathbb{N}\to\mathbb{N}} (p(\varepsilon_k p) \geq \varepsilon_k p \wedge c(p(\varepsilon_k p)) = k)$$

where k and p only need to be "good" at point $\varepsilon_k p$

Question: How to witness k and p given c and ε?

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Hence, for k and p as above,

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To witness α given ε, ω, q simply take

$$\begin{array}{rcl} \alpha & = & \mathsf{EPS}^{\omega}_{[\,]}(\varepsilon_i)(q) \\ s & = & [\alpha](\omega\alpha) \\ p & = & p_s \end{array}$$

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Spector'62 first defines **general bar recursion**:

6.2. *Bar recursion.* For ease in reading we omit showing G, H, Y as arguments of ϕ.

$$\phi(x, C) = \begin{cases} G(x, \langle C0, \cdots, C(x-1)\rangle) \text{ if } Y(\langle C0, \cdots, C(x-1)\rangle) < x\ , \\ H[\lambda a\phi(x', \langle C0, \cdots, C(x-1), a\rangle), x, \langle C0, \cdots, C(x-1)\rangle] \text{ otherwise.} \end{cases}$$

Thus $\phi(x, C)$ is defined outright if $Y(\langle C0, \cdots, C(x-1)\rangle) < x$, and in terms of $\lambda a\phi(x', \langle C0, \cdots, C(x-1), a\rangle)$ otherwise.

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But only uses **restricted bar recursion**:

10. The interpretation of F is provable in Σ_4. This is the only point where we make use of bar recursion, and we use it in the following restricted form, where the parameter G_0 is not exhibited as an argument of ϕ for greater readability.

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€100 Question: Quantifiers vs Selection Functions

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Spector general form is **iterated product of quantifiers**

$$\mathsf{EPQ}^{\omega}_{s}(\phi) \overset{K_R X^*}{=} \begin{cases} \lambda q.q([\,]) & \text{if } \omega(\hat{s}) < |s| \\ \phi_s \otimes^{\mathsf{q}} \lambda x.\mathsf{EPQ}^{\omega}_{s*x}(\phi) & \text{otherwise} \end{cases}$$

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Q: EPS is T-definable in EPQ, how about the converse?

References

M. Escardó and P. Oliva
Selection functions, bar recursion and backward induction
MSCS, 20(2):127-168, 2010

M. Escardó and P. Oliva
Computational interpretations of analysis via products of selection functions
CiE 2010, LNCS 6158, 2010

M. Escardó and P. Oliva
Sequential games and optimal strategies
Proceedings of the Royal Society A, 2011

P. Oliva and T. Powell
A finitisation of the infinite Ramsey theorem
In preparation, 2011