

Gödel's dialectica interpretation

Classical logic, arithmetic and analysis

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CLMPS – Mathematical Logic Session

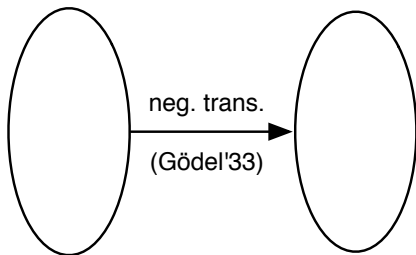
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21 July 2011



**Classical
Arithmetic**

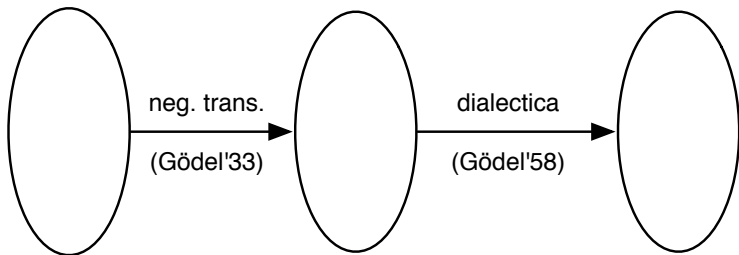
**Intuitionistic
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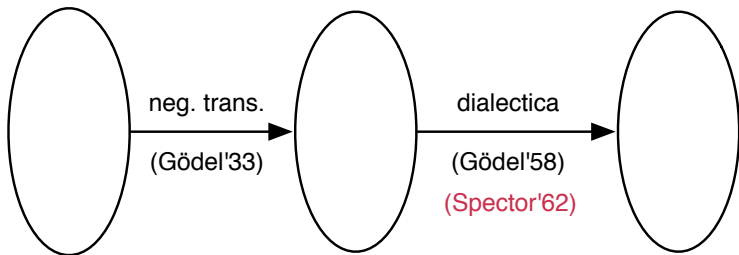
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**Classical
Arithmetic
Analysis**

**Intuitionistic
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**System T
+ bar recursion**



Outline

- 1 Motivation
 - A Challenge
- 2 The *dialectica* Interpretation
 - Classical Predicate Logic
- 3 Selection Functions
 - Interpreting Arithmetic
 - Interpreting Analysis
- 4 Challenge: Solution

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Inverse of a Function

Theorem

For any $H: X \rightarrow \mathbb{N}$ there exists $\alpha: \mathbb{N} \rightarrow X$ such that

$$H(\alpha k) = k \quad \text{whenever} \quad k \in \text{img}(H)$$

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$$\forall k(\exists x(Hx = k) \rightarrow \exists x'(Hx' = k))$$

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and invoke the **axiom of (countable) choice**

$$\exists \alpha \forall k(\exists x(Hx = k) \rightarrow H(\alpha k) = k)$$

No Injection from $\mathbb{N}^{\mathbb{N}}$ to \mathbb{N}

Corollary

For any $H: (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$ there exist $f, g: \mathbb{N} \rightarrow \mathbb{N}$ such that

$$f \neq g \quad \text{and} \quad H(f) \stackrel{\mathbb{N}}{=} H(g)$$

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(using **classical logic** and **countable choice**)



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Let $f_\alpha = \lambda n. \alpha(n)(n) + 1$ and $g_\alpha = \alpha(k_\alpha)$ where $k_\alpha = H(f_\alpha)$



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Clearly $f_\alpha(k_\alpha) \neq g_\alpha(k_\alpha)$ and $H(f_\alpha) = k_\alpha \stackrel{(*)}{=} H(g_\alpha)$ □

Interpreting Classical Theorems

How to “witness” a theorem like this:

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Q: *What does it mean to computationally interpret this?*

Herbrand Theorem

Theorem (Σ_1 -formulas)

If one proves $\exists x Q(x)$ classically then one can also prove

$$Q(t_0) \vee \dots \vee Q(t_n)$$

for a finite family of terms $(t_i)_{i \leq n}$.

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look at its Herbrand normal form

$$\forall p \exists x (Q_n(px) \rightarrow Q_n(x))$$

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Proof. Either $Q_n(p0)$ in which case we have (by weakening)

$$Q_n(p(p0)) \rightarrow Q_n(p0)$$

or $\neg Q_n(p0)$ in which case we have (by eq)

$$Q_n(p0) \rightarrow Q_n(0).$$

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Can even produce **single witness** if able to check $Q_n(x)$

$$\varepsilon_n p = \begin{cases} 0 & \text{if } \neg Q_n(p0) \\ p0 & \text{otherwise} \end{cases}$$

is such that $Q_n(p(\varepsilon_n p)) \rightarrow Q_n(\varepsilon_n p)$

Herbrand's Theorem

- Herbrand theorem only works for **prenex formulas**
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Gödel's *dialectica* interpretation!

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Map every formula to the $\exists\forall$ -fragment. For instance:

$$\exists x\forall yP(x, y) \quad \mapsto \quad \exists x \forall y P(x, y)$$

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$$\exists x P(x) \wedge \forall y Q(y) \quad \mapsto \quad \exists x \forall y (P(x) \wedge Q(y))$$

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$$\exists x P(x) \wedge \forall y Q(y) \quad \mapsto \quad \exists x \forall y (P(x) \wedge Q(y))$$

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Can think of the mapping

$$A \quad \mapsto \quad \exists x \forall y A_D(x, y)$$

as associating a set of **functionals** to each formula

$$A \quad \mapsto \quad W_A \equiv \{ t \in \mathbb{T} : \forall y A_D(t, y) \}$$

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Theorem (Soundness – Intuitionistic Version)

If A is HA-provable then W_A is non-empty.

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If A is HA-provable then W_A is non-empty. That is, if

*(1) A is provable in **Heyting arithmetic***

then

*(2) $A_D(t, y)$ is provable in the quantifier-free calculus T ,
for some term $t \in T$.*

Negative Translation

Extending dialectica interpretation to **classical** logic

Compose with embedding of CL into IL.

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Extending dialectica interpretation to **classical** logic

Compose with embedding of CL into IL. E.g.

$$(P)^N \quad \mapsto \quad P$$

$$(A \wedge B)^N \quad \mapsto \quad A^N \wedge B^N$$

$$(A \vee B)^N \quad \mapsto \quad A^N \vee B^N$$

$$(A \rightarrow B)^N \quad \mapsto \quad A^N \rightarrow B^N$$

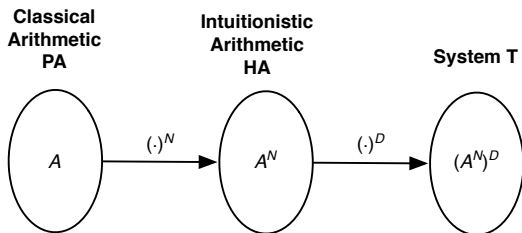
$$(\exists x A)^N \quad \mapsto \quad \exists x A^N$$

$$(\forall x A)^N \quad \mapsto \quad \forall x \neg\neg A^N$$

Then $\text{CL} \vdash A$ implies $\text{IL} \vdash \neg\neg A^N$

(Kuroda'51)

Soundness (Peano Arithmetic)



Theorem (Classical Version)

Assume A^N interpreted as $\exists x \forall y A_D^N(x, y)$. If

(1) A is provable in *Peano arithmetic*

then

(2) $A_D^N(t, y)$ is provable in the quantifier-free calculus T ,
for some term $t \in T$.

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Whose dialectica interpretation is

$$\exists \varepsilon_i \forall p (Q_i(p(\varepsilon_i p)) \rightarrow Q_i(\varepsilon_i p))$$

which has witness

$$\varepsilon_i p = \begin{cases} 0 & \text{if } \neg Q_i(p0) \\ p0 & \text{if } Q_i(p0) \end{cases}$$

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(Based on joint work with Martín Escardó)

Let

$$J_R X \equiv (X \rightarrow R) \rightarrow X$$

We think of the objects of type $J_R X$ as **selection functions**

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Consider:

- $R = \mathbb{B}$
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Also note that

$$\neg\neg\exists x^X \forall r^R Q(x, r) \quad \mapsto \quad \exists \varepsilon^{J_R X} \forall p^{X \rightarrow R} Q(\varepsilon p, p(\varepsilon p))$$

Binary Product of Selection Functions

Definition

Given $\varepsilon: J_R X$ and $\delta: J_R Y$ define their product

$$(\varepsilon \otimes \delta): J_R(X \times Y)$$

as

$$(\varepsilon \otimes \delta)(q^{X \times Y \rightarrow R}) \stackrel{X \times Y}{=} (a, b(a))$$

where $a \stackrel{X}{=} \varepsilon(\lambda x. q(x, b(x)))$ and $b(x) \stackrel{Y}{=} \delta(\lambda y. q(x, y))$.

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Let $J_R A \equiv (A \rightarrow R) \rightarrow A$ (as formula constructor)

Construction is essentially minimal logic proof of

$$J_R A \wedge J_R B \rightarrow J_R(A \wedge B)$$

Theorem on Finite Product of Selection Functions

Given sequence $\varepsilon : \prod_{i \leq n} J_R X_i$, define (\otimes prod of sel. fct.)

$$\left(\bigotimes_{i=0}^n \varepsilon_i \right) = \varepsilon_0 \otimes \dots \otimes \varepsilon_n \quad : J_R \prod_{i \leq n} X_i$$

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Theorem

Let $s = (\bigotimes_{i=0}^n \varepsilon_i)(q)$ with $q: \prod_{i=0}^n X_i \rightarrow R$. For $0 \leq i \leq n$

$$s_i \stackrel{X_i}{=} \varepsilon_i p_i$$

$$q_s \stackrel{R}{=} p_i(\varepsilon_i p_i)$$

for some $p_i: X_i \rightarrow R$.

Classical Arithmetic

We can prove (classically)

$$(+) \quad \forall i \leq n \exists x \forall y \underbrace{(Q_i(y) \rightarrow Q_i(x))}_{A_i(x,y)}$$

By finite choice (i.e. induction) we obtain

$$\exists s \forall i \leq n \forall y A_i(s_i, y)$$

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product of sel. fcts.

Classical Arithmetic

From theorem on product of selection functions we have:

$$s_i \stackrel{X_i}{=} \varepsilon_i p_i$$
$$qs \stackrel{R}{=} p_i(\varepsilon_i p_i)$$

In order to produce s such that

$$\forall i \leq n \quad \underbrace{A_i(s_i, qs)}_{Q_i(qs) \rightarrow Q_i(s_i)}$$

we only need to find ε_i such that for all p_i

$$\forall i \leq n \quad A_i(\varepsilon_i p_i, p_i(\varepsilon_i p_i))$$

(which is easy, as we have seen!)

Classical Analysis

What about infinitely many “uses” of classical logic? Given

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$$\exists \alpha \forall n \forall y \underbrace{(Q_n(y) \rightarrow Q_n(\alpha(n)))}_{A_n(\alpha(n),y)}$$

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$$\exists \alpha \forall n \forall y \underbrace{(Q_n(y) \rightarrow Q_n(\alpha(n)))}_{A_n(\alpha(n),y)}$$

whose *dialectica* interpretation (of negative translation) is

$$\forall \psi \forall q \exists \alpha \forall n \leq \psi \alpha \underbrace{(Q_n(q\alpha) \rightarrow Q_n(\alpha(n)))}_{A_n(\alpha(n),q\alpha)}$$

where $\psi: X^\omega \rightarrow \mathbb{N}$ and $q: X^\omega \rightarrow X$.

Controlled Iterated Product

This can be solved by a “controlled” iterated product

$$\left(\begin{array}{c} \psi \\ \otimes \\ \varepsilon \\ s \end{array} \right) (q) \stackrel{R}{=} \begin{cases} \mathbf{0} & \psi(\hat{s}) < |s| \\ \left(\varepsilon_{|s|} \otimes \lambda x^{X_{|s|}} \cdot \left(\begin{array}{c} \psi \\ \otimes \\ \varepsilon \\ s * x \end{array} \right) \right) (q) & \text{otherwise} \end{cases}$$

Theorem

Let $\alpha \stackrel{X^\omega}{=} \left(\begin{array}{c} \psi \\ \otimes \\ \varepsilon \\ \langle \rangle \end{array} \right) (q)$. There exist $p_i: X_i \rightarrow R$ s.t.

$$\alpha(i) \stackrel{X_i}{=} \varepsilon_i(p_i)$$

$$q\alpha \stackrel{R}{=} p_i(\varepsilon_i p_i)$$

for all $i \leq \psi(\alpha)$.

Outline

- 1 Motivation
 - A Challenge
- 2 The *dialectica* Interpretation
 - Classical Predicate Logic
- 3 Selection Functions
 - Interpreting Arithmetic
 - Interpreting Analysis
- 4 Challenge: Solution

Back to $(\mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N})$ -Example (I)

Corollary

For any $H: (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$ there exist $f, g: \mathbb{N} \rightarrow \mathbb{N}$ such that

$$f \neq g \quad \text{and} \quad H(f) \stackrel{\mathbb{N}}{=} H(g)$$

Proof.

Let $\alpha^{\mathbb{N} \rightarrow \mathbb{N}^{\mathbb{N}}}$ be some inverse of H , i.e. for all f and k

$$(*) \quad H(\alpha(k)) = k \quad \text{if} \quad H(f) = k$$

(using **classical logic** and **countable choice**)

Let $f_\alpha = \lambda n. \alpha(n)(n) + 1$ and $g_\alpha = \alpha(k_\alpha)$ where $k_\alpha = H(f_\alpha)$

Clearly $f_\alpha(k_\alpha) \neq g_\alpha(k_\alpha)$ and $H(f_\alpha) = k_\alpha \stackrel{(*)}{=} H(g_\alpha)$ □

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Construct approximation to inverse of H , i.e. $\alpha^{\mathbb{N} \rightarrow \mathbb{N}^{\mathbb{N}}}$ s.t.

$$\forall k \leq H(f_\alpha) \left(\underbrace{H(f_\alpha) = k \rightarrow H(\alpha(k)) = k}_{A_k(\alpha(k), f_\alpha)} \right)$$

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Enough to produce ε_k such that for all p

$$\underbrace{H(p(\varepsilon_k p)) = k \rightarrow H(\varepsilon_k p) = k}_{A_k(\varepsilon_k p, p(\varepsilon_k p))}$$

We have just built such ε_k 's!

Back to $(\mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N})$ -Example (II)

Let ε_i as before and $f_\alpha := \lambda n. \alpha(n)(n) + 1$

Theorem

Fix $H: \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}$. Let $q\alpha = f_\alpha$ and $\psi\alpha = H(f_\alpha)$. Define

$$\alpha = \left(\begin{array}{c} \psi \\ \otimes \varepsilon \\ \langle \rangle \end{array} \right) (q)$$

and $f = f_\alpha$ and $g = \alpha(\psi\alpha)$. Then

$$Hf = Hg \quad \text{and} \quad f(\psi\alpha) \neq g(\psi\alpha)$$

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