The *dialectica* Interpretation of Classical Logic

Wessex Seminar

Paulo Oliva

Swansea

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Outline

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1. Motivation
2. The *dialectica* Interpretation
3. Interpretation at Work
   - Classical Predicate Logic
   - Classical Arithmetic
   - Classical Analysis

No injection from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}$

Theorem

For any $H \colon (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ *there exist* $f, g \colon \mathbb{N} \to \mathbb{N}$ *such that*

$$f \neq g \quad \textit{and} \quad H(f) \stackrel{\mathbb{N}}{=} H(g)$$

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Proof.

Choose some inverse of H, i.e. $\alpha^{\mathbb{N} \to \mathbb{N}^{\mathbb{N}}}$ such that

$$\exists f (H(f) = k) \to H(\alpha(k)) = k \qquad (*)$$

(using **classical logic** and **axiom of choice**)

□

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(using **classical logic** and **axiom of choice**)

Let $f_\alpha = \lambda n.\alpha(n)(n) + 1$ and $g_\alpha = \alpha(k_\alpha)$ where $k_\alpha = H(f_\alpha)$

Clearly $f_\alpha(k_\alpha) \neq g_\alpha(k_\alpha)$ and $H(f_\alpha) = k_\alpha \overset{(*)}{=} H(g_\alpha)$ □

Interpreting Classical Theorems

We have used

$$\exists x \forall y (\underbrace{f_n y = 0 \to f_n x = 0}_{A_n(x,y)})$$

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Q: *What does it mean to computationally interpret this?*

Herbrand Theorem

Theorem (Σ_1-formulas)

If ones proves $\exists x P(x)$ classically then one can also prove

$$P(t_0) \vee \ldots \vee P(t_n)$$

for a finite family of terms $(t_i)_{i \leq n}$.

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If ones proves $\exists x \forall y P(x, y)$ *classically then one can also prove*

$$P(t_0, p(t_0)) \vee \ldots \vee P(t_n, p(t_n))$$

for a finite family of terms $(t_i)_{i \leq n}$ *built from* p.

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Back to Example

In our example

$$\exists x \forall y (\underbrace{f_n y = 0 \to f_n x = 0}_{A_n(x,y)})$$

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Can even produce **single witness** if able to check $f_n x = 0$

$$\varepsilon_n p = \begin{cases} p0 & \text{if } f_n(p0) = 0 \\ 0 & \text{otherwise} \end{cases}$$

is such that $A_n(\varepsilon_n p, p(\varepsilon_n p))$

Herbrand's Theorem

- Herbrand theorem only works for **prenex formulas**

- **Not modular** (as cut elimination)

  witnesses for A *and* $A \to B$ *doesn't give one for* B

- Similar to Kreisel's n.c.i. (which has same problems)

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- A modular generalisation that works for all formulas:

  Gödel's *dialectica* interpretation!

Outline

Gödel's *dialectica* Interpretation

Map every formula A to the $\exists\forall$-fragment

$$\exists x \forall y P(x,y) \qquad \mapsto \quad \exists x \; \forall y \; P(x,y)$$

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$$\exists x \forall y P(x,y) \quad \mapsto \quad \exists x\ \forall y\ P(x,y)$$

$$\forall x \exists y P(x,y) \quad \mapsto \quad \exists f\ \forall x\ P(x, fx)$$

Gödel's *dialectica* Interpretation

Map every formula A to the $\exists\forall$-fragment

$$\begin{aligned} \exists x \forall y P(x,y) &\quad\mapsto\quad \exists x\ \forall y\ P(x,y) \\ \forall x \exists y P(x,y) &\quad\mapsto\quad \exists f\ \forall x\ P(x, fx) \end{aligned}$$

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Map every formula A to the $\exists\forall$-fragment

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\forall x \exists y P(x,y) \quad &\mapsto \quad \exists f\ \forall x\ P(x, fx) \\
\exists x P(x) \wedge \forall y Q(y) \quad &\mapsto \quad \exists x\ \forall y\ (P(x) \wedge Q(y))
\end{aligned}$$

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\begin{array}{lcl}
\exists x \forall y P(x,y) & \mapsto & \exists x \ \forall y \ P(x,y) \\
\forall x \exists y P(x,y) & \mapsto & \exists f \ \forall x \ P(x, fx) \\
\exists x P(x) \wedge \forall y Q(y) & \mapsto & \exists x \ \forall y \ (P(x) \wedge Q(y)) \\
\exists x P(x) \to \exists y Q(y) & \mapsto & \exists f \ \forall x \ (P(x) \to Q(fx))
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\forall x P(x) \to \forall y Q(y) & \mapsto & \exists g\ \forall y\ (P(gy) \to Q(y)) \\
\neg \exists x \forall y P(x,y) & \mapsto & \exists p\ \forall x\ \neg P(x, px)
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\exists x P(x) \land \forall y Q(y) & \mapsto & \exists x\ \forall y\ (P(x) \land Q(y)) \\
\exists x P(x) \to \exists y Q(y) & \mapsto & \exists f\ \forall x\ (P(x) \to Q(fx)) \\
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\neg\neg \exists x \forall y P(x,y) &\mapsto \exists \varepsilon\ \forall p\ \neg\neg P(\varepsilon p, p(\varepsilon p))
\end{aligned}
$$

Soundness (Heyting Arithmetic)

Theorem (Intuitionistic Version)

Assume A *interpreted as* $\exists x \forall y A_D(x, y)$. *If*

(1) A *is provable in Heyting arithmetic*

then

(2) $A_D(t, y)$ *is provable in the quantifier-free calculus T,*

for some term t.

Negative Translation

Extending dialectica interpretation to **classical** logic

Compose with embedding of **classical** logic into IL.

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Extending dialectica interpretation to **classical** logic

Compose with embedding of **classical** logic into IL. E.g.

$$
\begin{array}{lcl}
(P)^N & \mapsto & P \\
(A \wedge B)^N & \mapsto & A^N \wedge B^N \\
(A \vee B)^N & \mapsto & A^N \vee B^N \\
(A \to B)^N & \mapsto & A^N \to B^N \\
(\exists x A)^N & \mapsto & \exists x A^N \\
(\forall x A)^N & \mapsto & \forall x \neg\neg A^N
\end{array}
$$

Then CL $\vdash A$ implies IL $\vdash \neg\neg A^N$ (Kuroda'51)

Soundness (Peano Arithmetic)

Theorem (Classical Version)

Assume A^N interpreted as $\exists x \forall y A_D^N(x, y)$. If

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Corollary (Gerhardy/Kohlenbach'2003)

Herbrand's theorem!

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We can prove (classically)

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Whose dialectica interpretation is

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Whose dialectica interpretation is

$$\exists \varepsilon_i \forall p A_i(\varepsilon_i p, p(\varepsilon_i p))$$

Hence, we have $A_i(\varepsilon_i p, p(\varepsilon_i p))$ for

$$\varepsilon_i p = \begin{cases} 0 & \text{if} \quad A_i(0, p0) \\ p0 & \text{if} \neg A_i(0, p0) \quad (\text{i.e. } f_i(p0) = 0 \wedge f_i(0) \neq 0) \end{cases}$$

Classical Arithmetic

We have

$$\forall i \leq n \exists x \forall y (\underbrace{f_i y = 0 \rightarrow f_i x = 0}_{A_i(x,y)})$$

By finite choice (i.e. induction) we obtain

$$\exists s \forall i \leq n \forall y A_i(s_i, y)$$

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ε_i as in previous slide

Claim: Can simply take $s = \left(\bigotimes_{i=0}^{n} \varepsilon_i\right)(q)$

Selection functions

Let $J_R X = (X \to R) \to X$

$\varepsilon \colon J_R X$ are called **selection functions**

Given sequence $\varepsilon \colon \Pi_{i \leq n} J_R X_i$, define (see Martin's talk)

$$\left(\bigotimes_{i=0}^{n} \varepsilon_i \right) : J_R \Pi_{i \leq n} X_i$$

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Theorem

Let $s = \left(\bigotimes_{i=0}^{n} \varepsilon_i\right)(q)$. *There exist* $p_i \colon X_i \to R$ *s.t.*

$$s_i \stackrel{X_i}{=} \varepsilon_i p_i$$

$$qs \stackrel{R}{=} p_i(\varepsilon_i p_i)$$

for $0 \leq i \leq n$.

Back to Example

Hence, given that

$$\begin{aligned} s_i &\stackrel{X}{=} \varepsilon_i p_i \\ qs &\stackrel{R}{=} p_i(\varepsilon_i p_i) \end{aligned}$$

In order to produce s such that

$$\forall i \leq n \qquad \underbrace{A_i(s_i, qs)}_{f_i(qs)=0 \,\rightarrow\, f_i s_i = 0}$$

we only need to find ε_i and p_i such that

$$\forall i \leq n \; A_i(\varepsilon_i p_i, p_i(\varepsilon_i p_i))$$

(which is easy, as we have seen!)

Another Example

Finite choice trivially implies **bounded collection**

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Bounded collection also directly proves the **infinite PHP**

$$\forall f^{\mathbb{N}\to[n]} \exists b \leq n \forall i \exists j \geq i (fj = b)$$

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dialectica interpretation of **IPHP**:

$$\forall f^{\mathbb{N} \to [n]} \forall \varepsilon \exists b \leq n \exists p (p(\varepsilon_b p) \geq \varepsilon_b p \wedge f(p(\varepsilon_b p)) = b)$$

Another Example

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dialectica interpretation of **IPHP**:

$$\forall f^{\mathbb{N}\to[n]} \forall \varepsilon \exists b \leq n \exists p (p(\varepsilon_b p) \geq \varepsilon_b p \wedge f(p(\varepsilon_b p)) = b)$$

which can be witnessed as

$$\boxed{b = f(\max((\bigotimes_{i=0}^{n} \varepsilon_i)(\max)))}$$

Classical Analysis

What about infinitely many "uses" of classical logic:

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Hence, by countable choice,

$$\exists \alpha \forall n \forall y (\underbrace{f_n y = 0 \to f_n(\alpha(n)) = 0}_{A_n(\alpha(n),y)})$$

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dialectica interpretation (of negative translation) is

$$\forall \psi \forall q \exists \alpha \forall n \leq \psi\alpha (\underbrace{f_n(q\alpha) = 0 \to f_n(\alpha(n)) = 0}_{A_n(\alpha(n),q\alpha)})$$

Selection functions (again)

This can be solved by a "controlled" iterated product

Theorem

Let $\alpha = \left(\bigotimes_{i=0}^{\psi} \varepsilon_i\right)(q)$. *There exist* $p_i \colon X_i \to R$ *s.t.*

$$\begin{aligned} \alpha_i &\overset{X}{=} \varepsilon_i(p_i) \\ q\alpha &\overset{R}{=} p_i(\varepsilon_i p_i) \end{aligned}$$

for all $i \leq \psi\alpha$.

Back to $(\mathbb{N}^{\mathbb{N}} \to \mathbb{N})$-Example (I)

Construct approximation to inverse of H, i.e. $\alpha^{\mathbb{N} \to \mathbb{N}^{\mathbb{N}}}$ st.

$$\forall i \leq \psi\alpha \left(\underbrace{H(q\alpha) = i \ \to \ H(\alpha(i)) = i}_{A_i(\alpha(i), q\alpha)} \right)$$

where

- $\psi\alpha \overset{\mathbb{N}}{=} H(\lambda n.\alpha(n)(n) + 1) = k_\alpha$
- $q\alpha = \lambda n.\alpha(n)(n) + 1$

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where

- $\psi\alpha \stackrel{\mathbb{N}}{=} H(\lambda n.\alpha(n)(n) + 1) = k_\alpha$
- $q\alpha = \lambda n.\alpha(n)(n) + 1$

Enough to produce ε_i and p_i such that

$$\underbrace{H(p_i(\varepsilon_i p_i)) = i \ \to \ H(\varepsilon_i p_i) = i}_{A_i(\varepsilon_i p_i, p_i(\varepsilon_i p_i))}$$

Back to $(\mathbb{N}^{\mathbb{N}} \to \mathbb{N})$-Example (II)

(cont)

We have just built such ε_i's

So, let q and ψ as in previous slide, and

$$\alpha = \left(\bigotimes_{i=0}^{\psi} \varepsilon_i \right) (q)$$

We have that

- $f_\alpha = \lambda n.\alpha(n)(n) + 1$
- $g_\alpha = \alpha(\psi(\alpha))$

effectively witnesses the theorem!

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