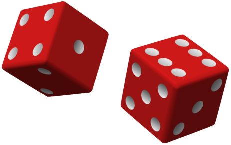
Sequential Games and Optimal Strategies

Paulo Oliva

(based on joint work with M. Escardó)

Queen Mary, University of London, UK



Loughborough University

Wednesday, 17 November 2010

Queen Mary – Theory Group

Queen Mary – Theory Group

- **Program verification and separation logic**
  B Cook, P O'Hearn, H Yang, D Distefano

Queen Mary – Theory Group

- **Program verification and separation logic**
  B Cook, P O'Hearn, H Yang, D Distefano
- **Verification of continuous dynamical systems**
  U Martin, P Oliva

Queen Mary – Theory Group

- **Program verification and separation logic**
  B Cook, P O'Hearn, H Yang, D Distefano
- **Verification of continuous dynamical systems**
  U Martin, P Oliva
- **Mathematical logic and proof theory**
  E Robinson, P Oliva

THE ROYAL SOCIETY

Queen Mary – Theory Group

- **Program verification and separation logic**
  B Cook, P O'Hearn, H Yang, D Distefano
- **Verification of continuous dynamical systems**
  U Martin, P Oliva
- **Mathematical logic and proof theory**
  E Robinson, P Oliva
- **Concurrency, complexity theory, information theory**
  K Honda, S Riis, P Malacaria

THE ROYAL SOCIETY

Outline

1. Game Theory
2. Quantifiers and Selection Functions
3. Generalisation

Outline

1. Game Theory
2. Quantifiers and Selection Functions
3. Generalisation

Game Theory

- Early development in the 19th century
- Formal approach with von Neumann (1930's)

John von Neumann

Game Theory

- Early development in the 19th century
- Formal approach with von Neumann (1930's)

- n players
- n strategy sets $X_1, \ldots, X_n$
- payoff function $q\colon \vec{X} \to \mathbb{R}^n$

John von Neumann

Game Theory

- Early development in the 19th century
- Formal approach with von Neumann (1930's)

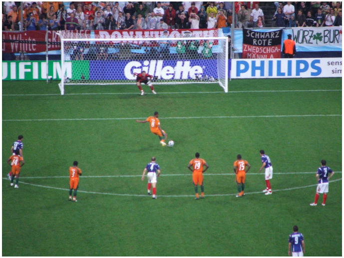
- n players
- n strategy sets $X_1, \ldots, X_n$
- payoff function $q\colon \vec{X} \to \mathbb{R}^n$

John von Neumann

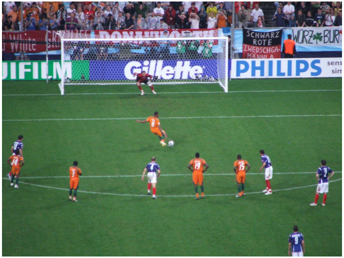
How should players choose their strategies

in order to maximise their individual payoffs?

Game Theory



Game Theory



Penalties

Two players

Strategy sets $X_1 = X_2 = \{L, R\}$

Payoff function

f	L	R
L	$(1, 0)$	$(0, 1)$
R	$(0, 1)$	$(1, 0)$

Game Theory

- No **winning** strategy!
- What about strategies in **equilibrium**?

Game Theory

- No **winning** strategy!
- What about strategies in **equilibrium**?

Definition (Nash)

Strategy profile $\vec{x}$ is in equilibrium if no player has an incentive to unilaterally change his strategy.

Game Theory

- No **winning** strategy!
- What about strategies in **equilibrium**?

Definition (Nash)

Strategy profile $\vec{x}$ is in equilibrium if no player has an incentive to unilaterally change his strategy.

The "penalty" example shows that strategy profiles in equilibrium not necessarily exist either.

Game Theory

- What if players choose "mixed" strategies
  i.e. player chooses probability distribution on strategies

Game Theory

- What if players choose "mixed" strategies
  i.e. player chooses probability distribution on strategies

Theorem (Nash)

Mixed strategies in equilibrium always exist.

Game Theory

- What if players choose "mixed" strategies
  i.e. player chooses probability distribution on strategies

Theorem (Nash)

Mixed strategies in equilibrium always exist.

The "penalty" example is again an illustration of this:
Players randomly choosing left or right is best they can do.

Simultaneous versus Sequential Games

- That's all in the case of **simultaneous** games
- With **sequential** games things are simpler and nicer

- Strategies: mappings from previous moves to current move
- Similar definition of Nash equilibrium

Simultaneous versus Sequential Games

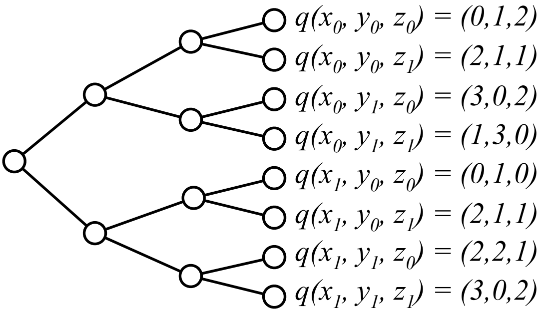
- That's all in the case of **simultaneous** games
- With **sequential** games things are simpler and nicer

- Strategies: mappings from previous moves to current move
- Similar definition of Nash equilibrium

> But equilibrium always exists and can be computed
> by a technique called **backward induction**

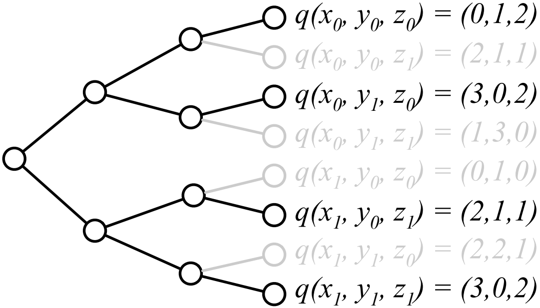
Backward Induction

$$q\colon X \times Y \times Z \to \mathbb{R}^3$$



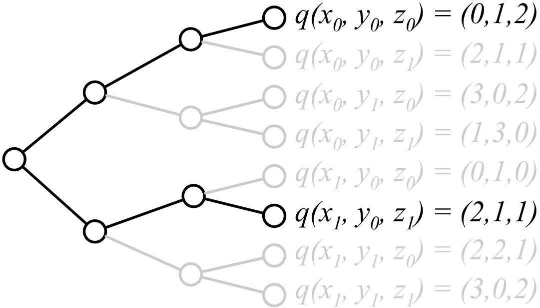
Backward Induction

$$q\colon X \times Y \times Z \to \mathbb{R}^3$$



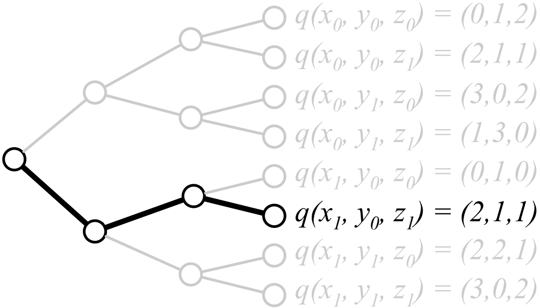
Backward Induction

$$q \colon X \times Y \times Z \to \mathbb{R}^3$$



Backward Induction

$$q\colon X \times Y \times Z \to \mathbb{R}^3$$

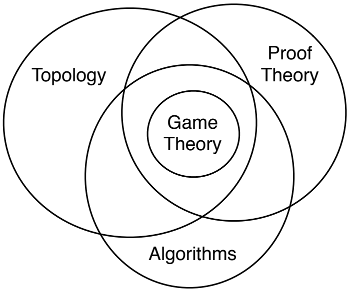


Our Recent Work

1. Generalised notions of sequential game, Nash equilibrium and backward induction

Our Recent Work

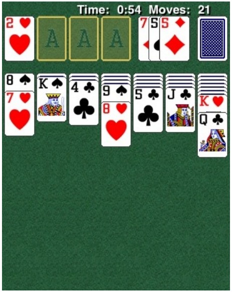
1. Generalised notions of sequential game, Nash equilibrium and backward induction
2. Showed how general notions appear in topology, proof theory, and algorithms, amongst others

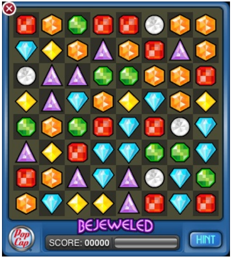


Outline

1. Game Theory
2. Quantifiers and Selection Functions
3. Generalisation

Single-player Games





Two-player Games

Two **players**: Black and White

Two-player Games

Two **players**: Black and White

Possible **outcomes**:

- Black wins
- White wins
- Draw

Two-player Games

Two **players**: Black and White

Possible **outcomes**:

- Black wins
- White wins
- Draw

Strategy: Choice of move at round k given previous moves

Another Game

Two **players**: John and Julia

Another Game

Two **players**: John and Julia

John splits a cake. Julia chooses one of the two pieces

Another Game

Two **players**: John and Julia

John splits a cake. Julia chooses one of the two pieces

Possible **outcomes**:

- John gets $N\%$ of the cake (John's payoff)
- Julia gets $(100 - N)\%$ of the cake (Julia's payoff)

Another Game

Two **players**: John and Julia

John splits a cake. Julia chooses one of the two pieces

Possible **outcomes**:

- John gets $N\%$ of the cake (John's payoff)
- Julia gets $(100 - N)\%$ of the cake (Julia's payoff)

Best strategy for John is to split cake into half

It is not a "winning strategy" but it is an **optimal strategy**

It maximises his payoff

Number of Player vs Number of Rounds

Number of players is not essential

It is important what the "goal" at each round is

Rounds with "**same goal**" mean played by "**same player**"

Number of Player vs Number of Rounds

Number of players is not essential

It is important what the “goal” at each round is

Rounds with “**same goal**” mean played by “**same player**”

How to describe the goal at a particular round?

Number of Player vs Number of Rounds

Number of players is not essential

It is important what the "goal" at each round is

Rounds with "**same goal**" mean played by "**same player**"

How to describe the goal at a particular round?

You could say: The goal is to win!

But maybe this is not possible (or might not even make sense)

Instead, the goal should be described as:

a choice of outcome from each set of possible outcomes

As in...

Q: How much would you like to pay for your flight?

As in...

Q: How much would you like to pay for your flight?

A: As little as possible!

Quantifiers

R = set of outcomes

X = set of possible moves

$$\boxed{\phi \in (X \to R) \to R}$$

describes the desired outcome $\phi p \in R$ given $p \in X \to R$

Quantifiers

R = set of outcomes

X = set of possible moves

$$\boxed{\phi \in (X \to R) \to R}$$

describes the desired outcome $\phi p \in R$ given $p \in X \to R$

In the example:

$$\begin{array}{lcl} R & = & \textit{prices (real numbers)} \\ X & = & \textit{possible flights} \\ X \to R & = & \textit{price of each flight} \\ \phi & = & \textit{minimal value functional} \end{array}$$

Quantifiers

$$\phi : \ (X \to R) \to R$$

Quantifiers

$$\phi : (X \to R) \to R$$

Other Examples

Operation	ϕ	:	$(X \to R) \to R$
Supremum	$\sup_{[0,1]}$	:	$([0,1] \to \mathbb{R}) \to \mathbb{R}$
Integration	$\int_0^1$	:	$([0,1] \to \mathbb{R}) \to \mathbb{R}$
Limit	$\lim$	:	$(\mathbb{N} \to R) \to R$
Quantifiers	$\forall_X, \exists_X$	:	$(X \to \mathbb{B}) \to \mathbb{B}$
Double negation	$\neg\neg X$	:	$(X \to \bot) \to \bot$
Fixed point operator	fix_X	:	$(X \to X) \to X$

Quantifiers

$$\phi : (X \to R) \to R \qquad (\equiv K_R X)$$

Other Examples

Operation	ϕ	:	$(X \to R) \to R$
Supremum	$\sup_{[0,1]}$	:	$([0,1] \to \mathbb{R}) \to \mathbb{R}$
Integration	$\int_0^1$	:	$([0,1] \to \mathbb{R}) \to \mathbb{R}$
Limit	$\lim$	:	$(\mathbb{N} \to R) \to R$
Quantifiers	$\forall_X, \exists_X$	:	$(X \to \mathbb{B}) \to \mathbb{B}$
Double negation	$\neg\neg X$	:	$(X \to \bot) \to \bot$
Fixed point operator	fix_X	:	$(X \to X) \to X$

Theorem (Maximum Value Theorem)

For any $p \in C[0,1]$ *there is a point* $a \in [0,1]$ *such that*

$$\sup p = p(a)$$

Theorem (Maximum Value Theorem)

For any $p \in C[0,1]$ *there is a point* $a \in [0,1]$ *such that*

$$\sup p = p(a)$$

Theorem (Mean Value Theorem)

For any $p \in C[0,1]$ *there is a point* $a \in [0,1]$ *such that*

$$\int_0^1 p = p(a)$$

Theorem (Witness Theorem)

For any $p\colon X \to \mathbb{B}$ *there is a point* $a \in X$ *such that*

$$\exists x^X p(x) \quad \Leftrightarrow \quad p(a)$$

(similar to Hilbert's ε*-term).*

Theorem (Witness Theorem)

For any $p\colon X \to \mathbb{B}$ *there is a point* $a \in X$ *such that*

$$\exists x^X p(x) \quad \Leftrightarrow \quad p(a)$$

(similar to Hilbert's ε*-term).*

Theorem (Counter-example Theorem)

For any $p\colon X \to \mathbb{B}$ *there is a point* $a \in X$ *such that*

$$\forall x^X p(x) \quad \Leftrightarrow \quad p(a)$$

(a *is counter-example to* p *if one exists).*

Let $JX \equiv (X \to R) \to X$.

Let $JX \equiv (X \to R) \to X$.

Definition (Selection Functions)

$\varepsilon \colon JX$ is called a **selection function** for $\phi \colon KX$ if

$$\phi(p) = p(\varepsilon p)$$

holds for all $p \colon X \to R$.

Let $JX \equiv (X \to R) \to X$.

Definition (Selection Functions)

$\varepsilon \colon JX$ is called a **selection function** for $\phi \colon KX$ if

$$\phi(p) = p(\varepsilon p)$$

holds for all $p \colon X \to R$.

Definition (Attainable Quantifiers)

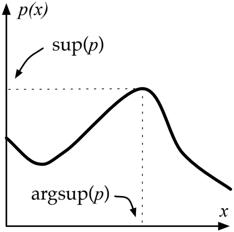
A quantifier $\phi \colon KX$ is called **attainable** if it has a selection function $\varepsilon \colon JX$.

For Instance

- $\sup\colon K_{\mathbb{R}}[0,1]$ is an attainable quantifier

$$\sup(p) = p(\text{argsup}(p))$$

where $\text{argsup}\colon J_{\mathbb{R}}[0,1]$.

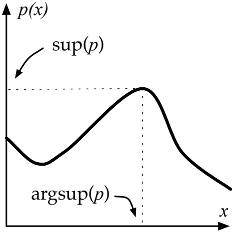


For Instance

- $\sup \colon K_{\mathbb{R}}[0,1]$ is an attainable quantifier

$$\sup(p) = p(\mathrm{argsup}(p))$$

where $\mathrm{argsup} \colon J_{\mathbb{R}}[0,1]$.

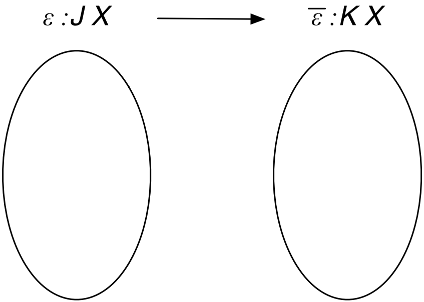


- $\mathsf{fix} \colon K_X X$ is an attainable quantifier

$$\mathsf{fix}(p) = p(\mathsf{fix}(p))$$

where $\mathsf{fix} \colon J_X X \ (= K_X X)$.

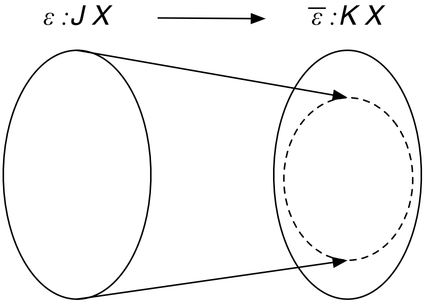
Selection Functions and Quantifiers



Every selection function $\varepsilon \colon JX$ defines a quantifier $\overline{\varepsilon} \colon KX$

$$\overline{\varepsilon}(p) \quad = \quad p(\varepsilon(p))$$

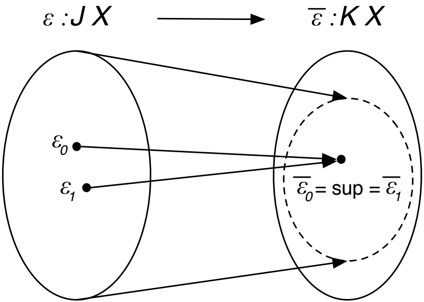
Selection Functions and Quantifiers



Not all quantifiers are attainable, e.g. $R = \{0, 1\}$

$$\phi(p) \quad = \quad 0$$

Selection Functions and Quantifiers



Different ε might define same ϕ, e.g. $X = [0,1]$ and $R = \mathbb{R}$

$$\begin{aligned} \varepsilon_0(p) &= \mu x . \sup p = p(x) \\ \varepsilon_1(p) &= \nu x . \sup p = p(x) \end{aligned}$$

Outline

1 Game Theory

2 Quantifiers and Selection Functions

3 Generalisation

Finite Sequential Games (n rounds)

Definition (A tuple $(R, (X_i)_{i<n}, (\phi_i)_{i<n}, q)$ where)

- R is the set of **possible outcomes**
- X_i is the set of **available moves** at round i
- $\phi_i \colon K_R X_i$ is the **goal quantifier** for round i
- $q \colon \Pi_{i=0}^{n-1} X_i \to R$ is the **outcome function**

Finite Sequential Games (n rounds)

Definition (A tuple $(R, (X_i)_{i<n}, (\phi_i)_{i<n}, q)$ where)

- R is the set of **possible outcomes**
- X_i is the set of **available moves** at round i
- $\phi_i \colon K_R X_i$ is the **goal quantifier** for round i
- $q \colon \Pi_{i=0}^{n-1} X_i \to R$ is the **outcome function**

Definition (Strategy)

Family of mappings

$$\text{next}_k \colon \Pi_{i=0}^{k-1} X_i \to X_k$$

Definition (Strategic Play)

Given strategy next_k and partial play $\vec{a} = a_0, \ldots, a_{k-1}$, the **strategic extension** of $\vec{a}$ is $\mathbf{b}^{\vec{a}} = b^{\vec{a}}_k, \ldots, b^{\vec{a}}_{n-1}$ where

$$b^{\vec{a}}_i = \mathsf{next}_i(\vec{a}, b^{\vec{a}}_k, \ldots, b^{\vec{a}}_{i-1}).$$

Definition (Strategic Play)

Given strategy next_k and partial play $\vec{a} = a_0, \ldots, a_{k-1}$, the **strategic extension** of $\vec{a}$ is $\mathbf{b}^{\vec{a}} = b_k^{\vec{a}}, \ldots, b_{n-1}^{\vec{a}}$ where

$$b_i^{\vec{a}} = \text{next}_i(\vec{a}, b_k^{\vec{a}}, \ldots, b_{i-1}^{\vec{a}}).$$

Definition (Optimal Strategy)

Strategy next_k is **optimal** if for any partial play $\vec{a}$

$$q(\vec{a}, \mathbf{b}^{\vec{a}}) = \phi_k(\lambda x_k.q(\vec{a}, x_k, \mathbf{b}^{\vec{a},x_k})).$$

Definition (Strategic Play)

Given strategy next_k and partial play $\vec{a} = a_0, \ldots, a_{k-1}$, the **strategic extension** of $\vec{a}$ is $\mathbf{b}^{\vec{a}} = b_k^{\vec{a}}, \ldots, b_{n-1}^{\vec{a}}$ where

$$b_i^{\vec{a}} = \mathsf{next}_i(\vec{a}, b_k^{\vec{a}}, \ldots, b_{i-1}^{\vec{a}}).$$

Definition (Optimal Strategy)

Strategy next_k is **optimal** if for any partial play $\vec{a}$

$$q(\vec{a}, \mathbf{b}^{\vec{a}}) = \phi_k(\lambda x_k . q(\vec{a}, x_k, \mathbf{b}^{\vec{a}, x_k})).$$

A product of selection functions computes optimal strategies

Standard Game Theory

When quantifiers are $\max$ or $\sup$ over finite or compact set

Then argsup exists (and hence $\sup$ is attainable)

Generalised Game	$\mapsto$	Standard Game
Optimal strategy	$\mapsto$	Strategy in Nash equilibrium
Product of argsup	$\mapsto$	Backward induction!

Fixed Point Theory

Fixed point operators are their own selection function

Generalised Game	$\mapsto$	Operators on product space
Optimal strategy	$\mapsto$	Bekiç's Lemma
Product of fix's	$\mapsto$	The proof!

Proof Theory

Proof interpretation

$$\exists i \leq n \forall x^{X_i} \exists r^R A_i(x, r) \quad \mapsto \quad \forall \varepsilon_{(\cdot)} \exists i \leq n \exists p A_i(\varepsilon_i p, p(\varepsilon_i p))$$

Proof Theory

Proof interpretation

$$\exists i \leq n \forall x^{X_i} \exists r^R A_i(x, r) \quad \mapsto \quad \underline{\forall \varepsilon_{(\cdot)}} \exists i \leq n \exists p A_i(\varepsilon_i p, p(\varepsilon_i p))$$

ε's define quantifiers, which partially define a game

Computational interpretation relies on completing the definition of the game so optimal strategy solves problem

Proof Theory

Proof interpretation

$$\exists i \leq n \forall x^{X_i} \exists r^R A_i(x,r) \quad \mapsto \quad \underline{\forall \varepsilon_{(\cdot)}} \exists i \leq n \exists p A_i(\varepsilon_i p, p(\varepsilon_i p))$$

ε's define quantifiers, which partially define a game

Computational interpretation relies on completing the definition of the game so optimal strategy solves problem

Existence of optimal strategy actually implies the consistency of mathematics!

Summary

- Generalised notion of sequential game
- Generalised notion of optimal strategy (equilibrium)
- Product of sel. fct. computes optimal strategies
- Results from fixed point theory, topology, proof theory, etc, can be viewed as optimal strategies in certain games

References

M. Escardó and P. Oliva
Selection functions, bar recursion and backward induction
MSCS, 20(2):127-168, 2010

M. Escardó and P. Oliva
What sequential games, the Tychnoff theorem and the double-negation shift have in common
ACM SIGPLAN MSFP, ACM Press 2010

M. Escardó and P. Oliva
Sequential games and optimal strategies
To appear: Proceedings of the Royal Society A, 2010