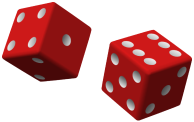
Theorems, Games, Proofs and Optimal Strategies

Paulo Oliva

(based on joint work with Martín Escardó)

Queen Mary, University of London, UK

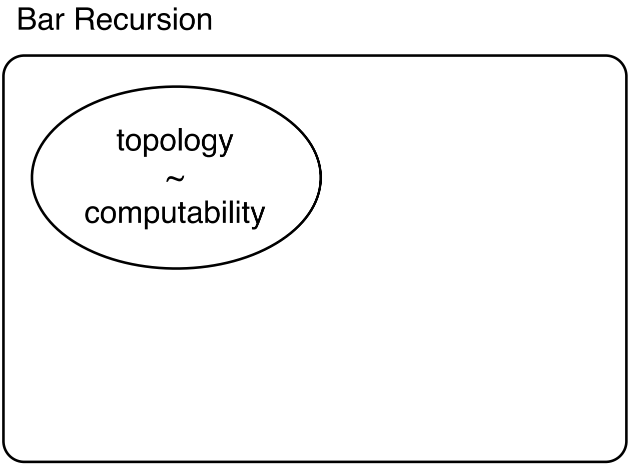


Program Extraction and Constructive Proofs

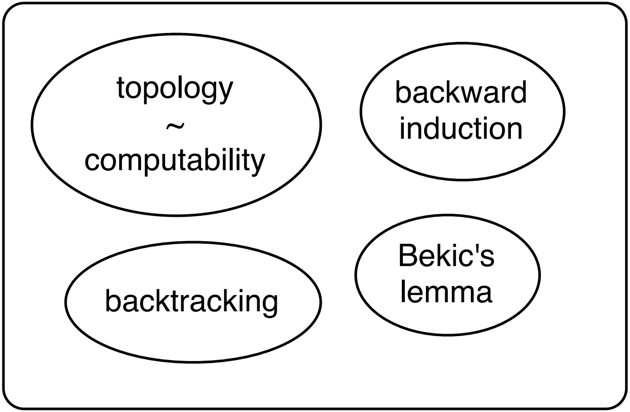
(in honour of Helmut Schwichtenberg)

Brno, 21 August 2010

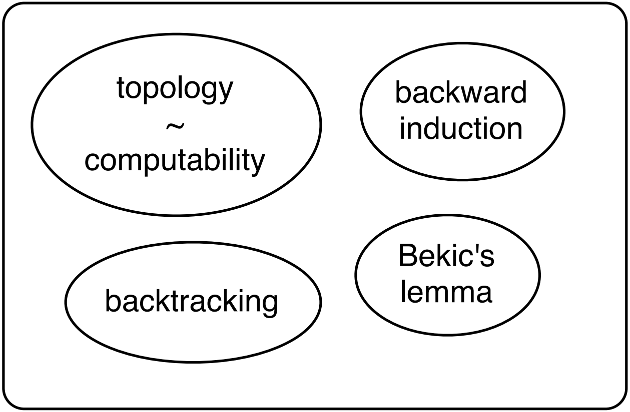
Bar Recursion


Bar Recursion
topology
~
computability

Bar Recursion



Bar Recursion ~ Optimal Strategies



Outline

Outline

Number of Player vs Number of Rounds

Number of players not essential

Goal at each round describes the game

Number of Player vs Number of Rounds

Number of players not essential

Goal at each round describes the game

Q: How much would you like to pay for your flight?

Number of Player vs Number of Rounds

Number of players not essential

Goal at each round describes the game

Q: How much would you like to pay for your flight?

A: As little as **possible**!

Number of Player vs Number of Rounds

Number of players not essential

Goal at each round describes the game

Q: How much would you like to pay for your flight?

A: As little as **possible**!

Q: What is Kasparov's aim at each round?

Number of Player vs Number of Rounds

Number of players not essential

Goal at each round describes the game

Q: How much would you like to pay for your flight?

A: As little as **possible**!

Q: What is Kasparov's aim at each round?

A: Capture the king, if **possible**, otherwise get a draw!

Number of Player vs Number of Rounds

Number of players not essential

Goal at each round describes the game

Q: How much would you like to pay for your flight?

A: As little as **possible**!

Q: What is Kasparov's aim at each round?

A: Capture the king, if **possible**, otherwise get a draw!

Goal: Choice of outcome given the **possible** outcomes

Target function

Describing Goal

X = set of possible moves

R = set of outcomes

Target function

$$\phi \in \underbrace{(X \to R) \to R}_{K_R X}$$

Target function

Describing Goal

X = set of possible moves

R = set of outcomes

Target function

$$\phi \in \underbrace{(X \to R) \to R}_{K_R X}$$

For instance

X = set of possible flights

$R = \mathbb{R}$ (prices of flight)

Target function

$$\inf \in (X \to \mathbb{R}) \to \mathbb{R}$$

Target function

Describing Goal

X = set of possible moves

R = set of outcomes

Target function

$$\phi \in \underbrace{(X \to R) \to R}_{K_R X}$$

For instance

X = set of possible flights

$R = \mathbb{R}$ (prices of flight)

Target function

$$\inf \in (X \to \mathbb{R}) \to \mathbb{R}$$

Other examples: $\exists, \forall, \sup, \int_0^1, \text{fix}, \ldots$

Target function – Product

As we compose quantifiers

$$\exists x^X \forall y^Y p(x, y)$$

we can compose target functions $\phi \colon K_R X$ and $\psi \colon K_R Y$ as

$$\boxed{(\phi \otimes \psi)(p) = \phi(\lambda x^X . \psi(\lambda y^Y . p(x, y)))}$$

Target function – Product

As we compose quantifiers

$$\exists x^X \forall y^Y p(x, y)$$

we can compose target functions $\phi\colon K_R X$ and $\psi\colon K_R Y$ as

$$\boxed{(\phi \otimes \psi)(p) = \phi(\lambda x^X . \psi(\lambda y^Y . p(x, y)))}$$

Target function – Product

As we compose quantifiers

$$\exists x^X \forall y^Y p(x, y)$$

we can compose target functions $\phi \colon K_R X$ and $\psi \colon K_R Y$ as

$$(\phi \otimes \psi)(p) = \phi(\lambda x^X . \psi(\lambda y^Y . p(x, y)))$$

Target function – Product

As we compose quantifiers

$$\exists x^X \forall y^Y p(x, y)$$

we can compose target functions $\phi \colon K_R X$ and $\psi \colon K_R Y$ as

$$(\phi \otimes \psi)(p) = \phi(\lambda x^X . \psi(\lambda y^Y . p(x, y)))$$

In general, given $\phi_i \colon K_R X_i$, for $0 \leq i \leq n$, define

$$\left(\bigotimes_{i=k}^{n} \phi_i \right) = \phi_k \otimes \left(\bigotimes_{i=k+1}^{n} \phi_i \right)$$

Sequential Games

A **sequential game with** n **rounds** is described by

- A set of **outcomes** R
- Sets of **available moves** X_i for each round $0 \leq i < n$
- **Target functions** $\phi_i \colon K_R X_i$ for each round $0 \leq i < n$
- An **outcome function** $q \colon \Pi_{i=0}^{n-1} X_i \to R$

Sequential Games

A **sequential game with** n **rounds** is described by

- A set of **outcomes** R
- Sets of **available moves** X_i for each round $0 \leq i < n$
- **Target functions** $\phi_i \colon K_R X_i$ for each round $0 \leq i < n$
- An **outcome function** $q \colon \Pi_{i=0}^{n-1} X_i \to R$

The **optimal outcome** is defined as

$$o = \left(\bigotimes_{i=0}^{n-1} \phi_i \right) (q)$$

Outline

Optimal Strategies

A **strategy** for the game is a family of mappings

$$\mathsf{next}_k : \Pi_{i=0}^{k-1} X_i \longrightarrow X_k$$

Optimal Strategies

A **strategy** for the game is a family of mappings

$$\mathsf{next}_k \colon \Pi_{i=0}^{k-1} X_i \to X_k$$

The **strategic extension** of partial play $\vec{a} = a_0, \ldots, a_{i-1}$ is

$$b_k^{\vec{a}} = \mathsf{next}_k(\vec{a}, b_i^{\vec{a}}, \ldots, b_{k-1}^{\vec{a}})$$

Optimal Strategies

A **strategy** for the game is a family of mappings

$$\text{next}_k : \Pi_{i=0}^{k-1} X_i \to X_k$$

The **strategic extension** of partial play $\vec{a} = a_0, \ldots, a_{i-1}$ is

$$b_k^{\vec{a}} = \text{next}_k(\vec{a}, b_i^{\vec{a}}, \ldots, b_{k-1}^{\vec{a}})$$

A strategy is **optimal** if for any partial play $\vec{a}$

$$\boxed{q(\vec{a}, \mathbf{b}^{\vec{a}}) = \phi_k(\lambda x_k . q(\vec{a}, x_k, \mathbf{b}^{\vec{a}, x_k}))}$$

Optimal Strategies

A **strategy** for the game is a family of mappings

$$\text{next}_k : \Pi_{i=0}^{k-1} X_i \rightarrow X_k$$

The **strategic extension** of partial play $\vec{a} = a_0, \ldots, a_{i-1}$ is

$$b_k^{\vec{a}} = \text{next}_k(\vec{a}, b_i^{\vec{a}}, \ldots, b_{k-1}^{\vec{a}})$$

A strategy is **optimal** if for any partial play $\vec{a}$

$$q(\vec{a}, \mathbf{b}^{\vec{a}}) = \phi_k(\lambda x_k . q(\vec{a}, x_k, \mathbf{b}^{\vec{a}, x_k}))$$

Optimal Strategies

A **strategy** for the game is a family of mappings

$$\mathsf{next}_k \colon \Pi_{i=0}^{k-1} X_i \to X_k$$

The **strategic extension** of partial play $\vec{a} = a_0, \ldots, a_{i-1}$ is

$$b_k^{\vec{a}} = \mathsf{next}_k(\vec{a}, b_i^{\vec{a}}, \ldots, b_{k-1}^{\vec{a}})$$

A strategy is **optimal** if for any partial play $\vec{a}$

$$\boxed{q(\vec{a}, \mathbf{b}^{\vec{a}}) = \phi_k(\lambda x_k . q(\vec{a}, x_k, \mathbf{b}^{\vec{a}, x_k}))}$$

Proposition. $q(\mathbf{b}) = o$

Selection Functions

$$J_R X \equiv (X \to R) \to X$$

Selection Functions

$$J_R X \equiv (X \to R) \to X$$

$\varepsilon \colon J_R X$ is called a **selection function** for $\phi \colon K_R X$ if

$$\boxed{\phi(p) = p(\varepsilon(p))}$$

holds for all $p \colon X \to R$.

Selection Functions

$$J_R X \equiv (X \to R) \to X$$

$\varepsilon\colon J_R X$ is called a **selection function** for $\phi\colon K_R X$ if

$$\boxed{\phi(p) = p(\varepsilon(p))}$$

holds for all $p\colon X \to R$.

$\phi\colon K_R X$ is **attainable** if it has a selection function $\varepsilon\colon J_R X$

Product of Selection Functions

Given $\varepsilon \colon J_R X$ and $\delta \colon J_R Y$ define

$$(\varepsilon \otimes \delta)(p^{X \times Y \to R}) \stackrel{X \times Y}{=} (a, b(a))$$

where

$$a = \varepsilon(\lambda x.p(x, b(x)))$$

$$b(x) = \delta(\lambda y.p(x, y))$$

Product of Selection Functions

Given $\varepsilon \colon J_R X$ and $\delta \colon J_R Y$ define

$$(\varepsilon \otimes \delta)(p^{X \times Y \to R}) \stackrel{X \times Y}{=} (a, b(a))$$

where
$$a = \varepsilon(\lambda x . p(x, b(x)))$$
$$b(x) = \delta(\lambda y . p(x, y))$$

Given $\varepsilon_i \colon J_R X_i$, for $0 \leq i \leq n$, define

$$\left(\bigotimes_{i=k}^{n} \varepsilon_i \right) = \varepsilon_k \otimes \left(\bigotimes_{i=k+1}^{n} \varepsilon_i \right)$$

Computing Optimal Strategies

If ϕ_i are attainable with selection functions ε_i then

$$\mathsf{next}_k(x_0, \ldots, x_{k-1}) \stackrel{X_k}{=} \left(\left(\bigotimes_{i=k}^{n-1} \varepsilon_i \right) (q_{x_0, \ldots, x_{k-1}}) \right)_0$$

is an **optimal strategy** for the game $(R, (X_i)_{i<n}, (\phi_i)_{i<n}, q)$

Computing Optimal Strategies

If ϕ_i are attainable with selection functions ε_i then

$$\mathsf{next}_k(x_0, \ldots, x_{k-1}) \stackrel{X_k}{=} \left(\left(\bigotimes_{i=k}^{n-1} \varepsilon_i\right)(q_{x_0,\ldots,x_{k-1}})\right)_0$$

is an **optimal strategy** for the game $(R, (X_i)_{i<n}, (\phi_i)_{i<n}, q)$

Moreover,

$$\vec{a} = \left(\bigotimes_{i=0}^{n-1} \varepsilon_i\right)(q)$$

is the **strategic (optimal) play**

Corollary

Let $\vec{a}$ be strategic play and o the optimal outcome.

There are functions $p_k \colon X_k \to R$ such that

$$
\boxed{\begin{array}{llll}
a_k & \stackrel{X_k}{=} & \varepsilon_k(p_k) & \quad (\textit{optimal move}) \\
o & \stackrel{R}{=} & \phi_k(p_k) & \quad (\textit{optimal outcome})
\end{array}}
$$

for all $0 \leq k < n$.

Corollary

Let $\vec{a}$ be strategic play and o the optimal outcome.

There are functions $p_k \colon X_k \to R$ such that

$$\begin{array}{llll} a_k & \stackrel{X_k}{=} & \varepsilon_k(p_k) & (\textit{optimal move}) \\ o & \stackrel{R}{=} & \phi_k(p_k) & (\textit{optimal outcome}) \end{array}$$

for all $0 \leq k < n$.

Proof. Take

$$p_k = \lambda x_k . \left(\bigotimes_{i=k+1}^{n-1} \phi_i \right) (q_{a_0,\ldots,a_{k-1},x_k})$$

Corollary

Let $\vec{a}$ be strategic play and o the optimal outcome.

There are functions $p_k \colon X_k \to R$ such that

$$a_k \overset{X_k}{=} \varepsilon_k(p_k) \qquad \textit{(optimal move)}$$

$$o \overset{R}{=} \phi_k(p_k) \qquad \textit{(optimal outcome)}$$

for all $0 \leq k < n$.

$\phi_k(p_k) \to p_k(\varepsilon_k(p_k))$

Proof. Take

$$p_k = \lambda x_k . \left(\bigotimes_{i=k+1}^{n-1} \phi_i \right) (q_{a_0, \ldots, a_{k-1}, x_k})$$

Outline

Functional interpretations

Dialectica

$$\underbrace{\exists x^X}_{\text{witness}} \forall y^R A_0(x, y) \quad \mapsto \quad \underbrace{\exists \varepsilon^{J_R X}}_{\text{sel. fct.}} \forall p^{X \to R} A_0(\varepsilon p, p(\varepsilon p))$$

Functional interpretations

Dialectica

$$\underbrace{\exists x^X}_{\text{witness}} \; \forall y^R A_0(x, y) \quad \mapsto \quad \underbrace{\exists \varepsilon^{J_R X}}_{\text{sel. fct.}} \; \forall p^{X \to R} A_0(\varepsilon p, p(\varepsilon p))$$

Realizability

Proofs of $\neg\neg A$ are *target functions*

Proofs of $\neg A \to A$ are *selection functions*

Dialectica interpretation – IPHP

Infinite pigeon-hole principle $(c \colon \mathbb{N} \to \{0, 1, \ldots, n\})$

$$\exists k \leq n \forall x \exists y (y \geq x \wedge c(y) = k)$$

Dialectica interpretation – IPHP

Infinite pigeon-hole principle $(c\colon \mathbb{N} \to \{0, 1, \ldots, n\})$

$$\exists k \leq n \forall x \exists y (y \geq x \wedge c(y) = k)$$

$$\exists k \leq n \exists p \forall x (px \geq x \wedge c(px) = k)$$

Dialectica interpretation – IPHP

Infinite pigeon-hole principle $(c\colon \mathbb{N} \to \{0, 1, \ldots, n\})$

$$\exists k \leq n \forall x \exists y (y \geq x \wedge c(y) = k)$$

$$\exists k \leq n \exists p \forall x (px \geq x \wedge c(px) = k)$$

$$\forall \varepsilon^{\Pi_{i \leq n} J_{\mathbb{N}} \mathbb{N}} \exists k \leq n \exists p (p(\varepsilon_k p) \geq \varepsilon_k p \wedge c(p(\varepsilon_k p)) = k)$$

Dialectica interpretation – IPHP

Infinite pigeon-hole principle $(c \colon \mathbb{N} \to \{0, 1, \ldots, n\})$

$$\exists k \leq n \forall x \exists y (y \geq x \wedge c(y) = k)$$

$$\exists k \leq n \exists p \forall x (px \geq x \wedge c(px) = k)$$

$$\forall \varepsilon^{\Pi_{i \leq n} J_{\mathbb{N}} \mathbb{N}} \exists k \leq n \exists p (p(\varepsilon_k p) \geq \varepsilon_k p \wedge c(p(\varepsilon_k p)) = k)$$

$$\forall \varepsilon^{\Pi_{i \leq n} J_{\mathbb{N}} \mathbb{N}} \exists k \leq n \exists q^{\Pi_{i \leq n} \mathbb{N} \to \mathbb{N}} (o \geq a_k \wedge c(o) = k)$$

Dialectica interpretation – IPHP

Infinite pigeon-hole principle $(c \colon \mathbb{N} \to \{0, 1, \ldots, n\})$

$$\exists k \leq n \forall x \exists y (y \geq x \wedge c(y) = k)$$

$$\exists k \leq n \exists p \forall x (px \geq x \wedge c(px) = k)$$

$$\forall \varepsilon^{\Pi_{i \leq n} J_{\mathbb{N}} \mathbb{N}} \exists k \leq n \exists p (p(\varepsilon_k p) \geq \varepsilon_k p \wedge c(p(\varepsilon_k p)) = k)$$

$$\forall \varepsilon^{\Pi_{i \leq n} J_{\mathbb{N}} \mathbb{N}} \exists k \leq n \exists q^{\Pi_{i \leq n} \mathbb{N} \to \mathbb{N}} (o \geq a_k \wedge c(o) = k)$$

Solution

$$\begin{aligned} q &= \max \\ k &= c(o) \end{aligned}$$

Realizability – DNS

Realizer (A-trans + **mr**) for $\mathsf{DNS_f}$

$$\forall k \leq n \neg\neg A_k \to \neg\neg \forall k \leq n A_k$$

is litterally the product of target functions

$$\bigotimes \quad \mathbf{mr} \quad \mathsf{DNS_f}$$

Realizability – DNS

Realizer (A-trans + **mr**) for $\mathbf{DNS}_{\mathsf{f}}$

$$\forall k \leq n \neg\neg A_k \to \neg\neg\forall k \leq n A_k$$

is litterally the product of target functions

$$\bigotimes \ \mathbf{mr} \ \mathbf{DNS}_{\mathsf{f}}$$

Unbounded **DNS** not realizable

Infinite product of target functions does exist (even in $\mathcal{C}$)

Realizability – DNS

Realizer (A-trans + **mr**) for **DNS**$_\mathsf{f}$

$$\forall k \leq n \neg\neg A_k \rightarrow \neg\neg\forall k \leq n A_k$$

is litterally the product of target functions

$$\bigotimes \quad \mathbf{mr} \quad \mathsf{DNS}_\mathsf{f}$$

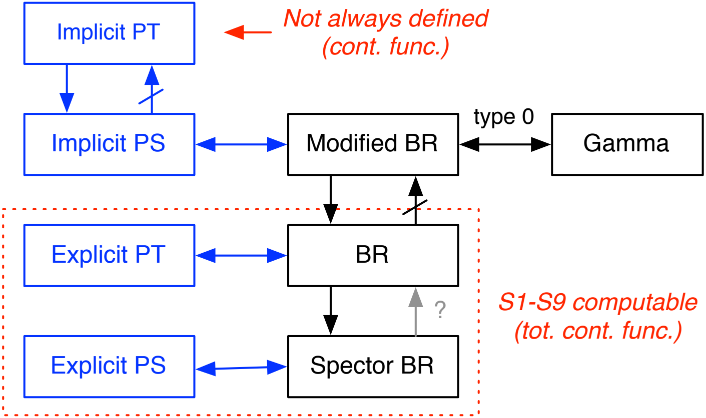
Unbounded **DNS** not realizable

Infinite product of target functions does exist (even in $\mathcal{C}$)

Infinite product of **selection functions** realizes

$$\forall k(\neg A_k \rightarrow A_k) \rightarrow (\neg\forall k A_k \rightarrow \forall k A_k)$$

Bar recursion



References

- M. Escardó and P. Oliva
  Selection functions, bar recursion and backward induction
  MSCS, 20(2):127-168, 2010

- M. Escardó and P. Oliva
  The Peirce translation and the double negation shift
  LNCS, CiE'2010

- M. Escardó and P. Oliva
  Computational interpretations of analysis via products of selection functions
  LNCS, CiE'2010

- M. Escardó and P. Oliva
  What sequential games, the Tychnoff theorem and the double-negation shift have in common
  MSFP 2010, ACM SIGPLAN