Sequential Games and Optimal Strategies

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(based on joint work with M. Escardó)

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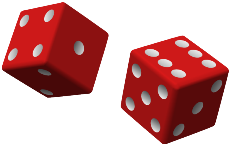
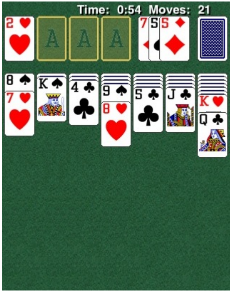
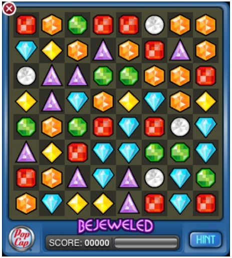


Logic Colloquium

Paris, 25 July 2010

Single-player Games





Two-player Games

Two **players**: Black and White

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Possible **outcomes**:

- Black wins
- White wins
- Draw

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Strategy: Choice of move at round k given previous moves

Another Game

Two **players**: John and Julia

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John splits a cake. Julia chooses one of the two pieces

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Possible **outcomes**:

- John gets $N\%$ of the cake (John's payoff)
- Julia gets $(100 - N)\%$ of the cake (Julia's payoff)

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Another Game

Two **players**: John and Julia

John splits a cake. Julia chooses one of the two pieces

Possible **outcomes**:

- John gets $N\%$ of the cake (John's payoff)
- Julia gets $(100 - N)\%$ of the cake (Julia's payoff)

Best strategy for John is to split cake into half

It is not a "winning strategy" but it is an **optimal strategy**

It maximises his payoff

Number of Player vs Number of Rounds

Number of players is not essential

It is important what the “goal” at each round is

Rounds with “**same goal**” mean played by “**same player**”

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How to describe the goal at a particular round?

You could say: The goal is to win!

But maybe this is not possible (or might not even make sense)

Instead, the goal should be described as

a choice of outcome from each set of possible outcomes

As in...

Q: How much would you like to play for your flight?

As in...

Q: How much would you like to play for your flight?

A: As little as possible!

Target function

If $R =$ set of outcomes and $X =$ set of possible moves then

$$\phi \in (X \to R) \to R$$

describes the desired outcome $\phi p \in R$ given that the outcome of the game $px \in R$ for each move $x \in X$ is given.

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In the example:

$$\begin{array}{lcl} X & = & \textit{possible flights} \\ R & = & \textit{real number} \\ X \to R & = & \textit{price of each flight} \\ \phi & = & \textit{minimal value functional} \end{array}$$

Outline

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Generalised quantifiers

$$\phi : (X \to R) \to R$$

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For instance

Operation	ϕ	:	$(X \to R) \to R$
Quantifiers	$\forall_X, \exists_X$	:	$(X \to \mathbb{B}) \to \mathbb{B}$
Double negation	$\neg\neg X$	:	$(X \to \bot) \to \bot$
Integration	$\int_0^1$	:	$([0,1] \to \mathbb{R}) \to \mathbb{R}$
Supremum	$\sup_{[0,1]}$	:	$([0,1] \to \mathbb{R}) \to \mathbb{R}$
Limit	$\lim$	:	$(\mathbb{N} \to R) \to R$
Fixed point operator	fix_X	:	$(X \to X) \to X$

Generalised quantifiers

$$\phi : (X \to R) \to R \qquad (\equiv K_R X)$$

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Definition (Product of Generalised Quantifiers)

Given $\phi \colon KX$ and $\psi \colon KY$ define $\phi \otimes \psi \ : \ K(X \times Y)$

$$(\phi \otimes \psi)(p) :\stackrel{R}{\equiv} \phi(\lambda x^X . \psi(\lambda y^Y . p(x,y)))$$

where $p \colon X \times Y \to R$.

Theorem (Mean Value Theorem)

For any $p \in C[0,1]$ *there is a point* $a \in [0,1]$ *such that*

$$\int_0^1 p = p(a)$$

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Theorem (Maximum Value Theorem)

For any $p \in C[0,1]$ *there is a point* $a \in [0,1]$ *such that*

$$\sup p = p(a)$$

Theorem (Witness Theorem)

For any $p\colon X \to \mathbb{B}$ there is a point $a \in X$ such that

$$\exists x^X p(x) \quad \Leftrightarrow \quad p(a)$$

(similar to Hilbert's ε-term).

Theorem (Witness Theorem)

For any $p\colon X \to \mathbb{B}$ there is a point $a \in X$ such that

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Theorem (Counter-example Theorem)

For any $p\colon X \to \mathbb{B}$ there is a point $a \in X$ such that

$$\forall x^X p(x) \quad \Leftrightarrow \quad p(a)$$

(a is counter-example to p if one exists).

Let $JX \equiv (X \to R) \to X$.

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Definition (Selection Functions)

$\varepsilon \colon JX$ is called a **selection function** for $\phi \colon KX$ if

$$\phi(p) = p(\varepsilon p)$$

holds for all $p \colon X \to R$.

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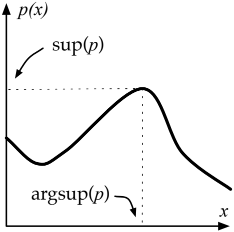
Definition (Attainable Quantifiers)

A generalised quantifier $\phi\colon KX$ is called **attainable** if it has a selection function $\varepsilon\colon JX$.

For Instance

- $\sup\colon K_{\mathbb{R}}[0,1]$ is an attainable quantifier as
$$\sup(p) = p(\operatorname{argsup}(p))$$
where $\operatorname{argsup}\colon J_{\mathbb{R}}[0,1]$.



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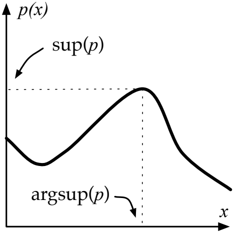
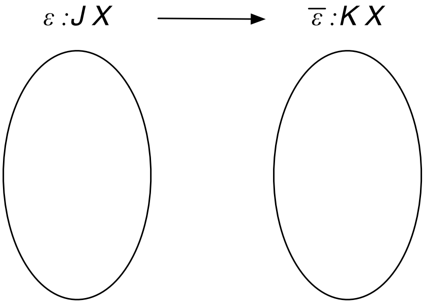


- $\mathsf{fix}\colon K_X X$ is an attainable quantifier as
$$\mathsf{fix}(p) = p(\mathsf{fix}(p))$$
where $\mathsf{fix}\colon J_X X \; (= K_X X)$.

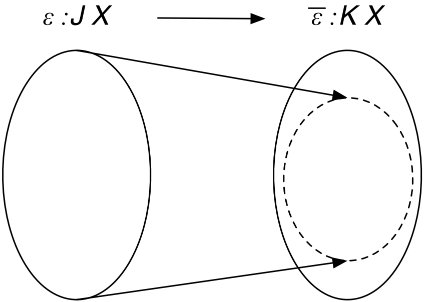
Selection Functions and Generalised Quantifiers



Every selection function $\varepsilon \colon JX$ defines a quantifier $\overline{\varepsilon} \colon KX$

$$\overline{\varepsilon}(p) \quad = \quad p(\varepsilon(p))$$

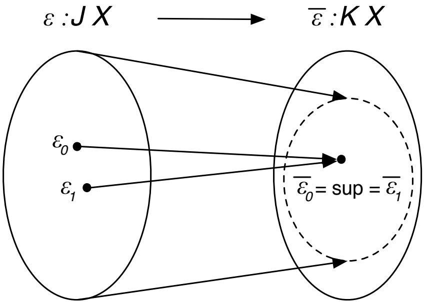
Selection Functions and Generalised Quantifiers



Not all quantifiers are attainable, e.g. $R = \{0, 1\}$

$$\phi(p) \quad = \quad 0$$

Selection Functions and Generalised Quantifiers



Different ε might define same ϕ, e.g. $X = [0,1]$ and $R = \mathbb{R}$

$$\begin{aligned}\varepsilon_0(p) &= \mu x . \sup p = p(x) \\ \varepsilon_1(p) &= \nu x . \sup p = p(x)\end{aligned}$$

Quantifier Elimination

Suppose

$$
\begin{aligned}
\exists x\, q(x) &= q(\varepsilon q) \\
\forall y\, q(y) &= q(\delta q).
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Quantifier Elimination

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Then

$$\exists x \forall y\, p(x, y) = \exists x\, p(x, b(x))$$

where

$$b(x) = \delta(\lambda y. p(x, y))$$

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$$\begin{aligned} b(x) &= \delta(\lambda y. p(x,y)) \\ a &= \varepsilon(\lambda x. p(x, b(x))). \end{aligned}$$

Quantifier Elimination

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Then

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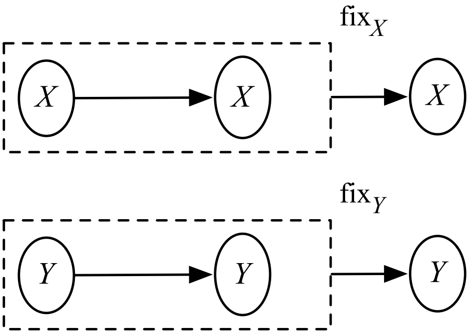
Bekič's Lemma

Lemma

If X and Y have fixed point operators then so does $X \times Y$.



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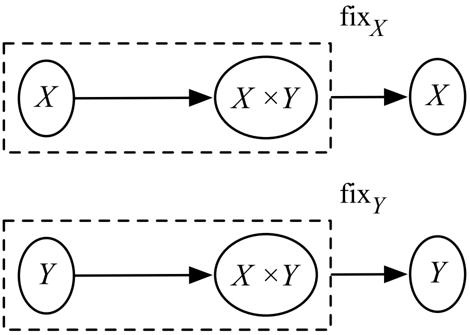

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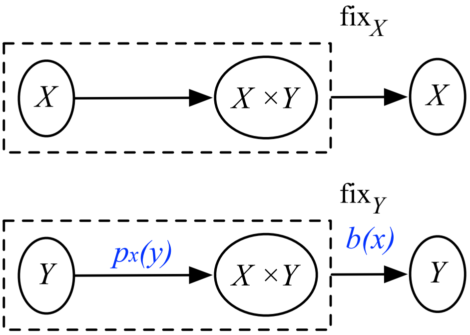

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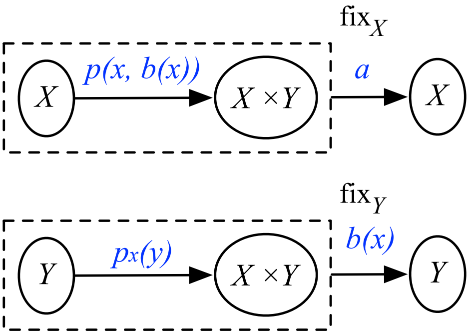

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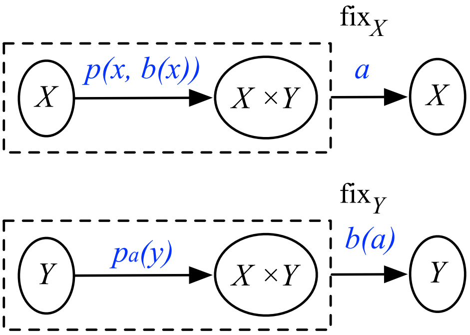

Definition (Product of Selection Functions)

Given $\varepsilon \colon JX$ and $\delta \colon JY$ define $\varepsilon \otimes \delta \colon J(X \times Y)$ as

$$(\varepsilon \otimes \delta)(p^{X \times Y \to R}) \stackrel{X \times Y}{:=} (\quad , \qquad)$$

where

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Theorem

$\overline{\varepsilon} \otimes \overline{\delta} = \overline{\varepsilon \otimes \delta}$

Iterated Product of Selection Functions

Finite iteration

$$\bigotimes_{i=k}^{n} \mathcal{E}_i \stackrel{J\Pi X_i}{=} \mathcal{E}_k \otimes \left(\bigotimes_{i=k+1}^{n} \mathcal{E}_i \right)$$

Iterated Product of Selection Functions

Infinite iteration $(R$ discrete, $R^{\Pi X_i}$ continuous$)$

$$\bigotimes_{i=k}^{\infty} \varepsilon_i \stackrel{J\Pi X_i}{=} \varepsilon_k \otimes \left(\bigotimes_{i=k+1}^{\infty} \varepsilon_i \right)$$

Iterated Product of Selection Functions

Infinite iteration I (R discrete, $R^{\Pi X_i}$ continuous)

$$\bigotimes_{i=k}^{\infty} \varepsilon_i \overset{J\Pi X_i}{=} \varepsilon_k \otimes \left(\bigotimes_{i=k+1}^{\infty} \varepsilon_i \right)$$

Infinite iteration II ($l \colon R \to \mathbb{N}$, $\mathbb{N}^{\Pi X_i}$ continuous)

$$\left(\bigotimes_{i=k}^{\infty} \varepsilon_i \right)(q) \overset{\Pi X_i}{=} \begin{cases} \mathbf{c} & \text{if } k < l(q(\mathbf{c})) \\ \left(\varepsilon_k \otimes \left(\bigotimes_{i=k+1}^{\infty} \varepsilon_i \right) \right)(q) & \text{otherwise} \end{cases}$$

Iterated Product of Selection Functions

Infinite iteration I (R discrete, $R^{\Pi X_i}$ continuous) = MBR

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Outline

Finite Games (n rounds)

Definition (A tuple $(R, (X_i)_{i<n}, (\phi_i)_{i<n}, q)$ where)

- R is the set of **possible outcomes**
- X_i is the set of **available moves** at round i
- $\phi_i \colon K_R X_i$ is the **goal quantifier** for round i
- $q \colon \Pi_{i=0}^{n-1} X_i \to R$ is the **outcome function**

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Definition (Strategy)

Family of mappings

$$\mathsf{next}_k \colon \Pi_{i=0}^{k-1} X_i \to X_k$$

Optimal Strategies

Definition (Strategic Play)

Given strategy next_k and partial play $\vec{a} = a_0, \ldots, a_{k-1}$, the **strategic extension** of $\vec{a}$ is $\mathbf{b}^{\vec{a}} = b_k^{\vec{a}}, \ldots, b_{n-1}^{\vec{a}}$ where

$$b_i^{\vec{a}} = \mathsf{next}_i(\vec{a}, b_k^{\vec{a}}, \ldots, b_{i-1}^{\vec{a}}).$$

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$$b_i^{\vec{a}} = \text{next}_i(\vec{a}, b_k^{\vec{a}}, \ldots, b_{i-1}^{\vec{a}}).$$

Definition (Optimal Strategy)

Strategy next_k is **optimal** if for any partial play $\vec{a}$

$$q(\vec{a}, \mathbf{b}^{\vec{a}}) = \phi_k(\lambda x_k.q(\vec{a}, x_k, \mathbf{b}^{\vec{a},x_k})).$$

Examples

Example (Nash Equilibrium with common payoff)

Moves X_i	Sets of moves
Outcomes R	Payoff $\mathbb{R}$
Goal quantifier ϕ_i	Maximal value function
Outcome function q	Payoff function $q\colon \Pi_{i=0}^{n-1} X_i \to \mathbb{R}$

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Optimal strategy

$$\mathsf{next}_k(x_0, \ldots, x_{k-1}) = \operatorname{argsup}_{x_k} \sup_{x_{k+1}} \ldots \sup_{x_{n-1}} q(\vec{x})$$

Examples

Example (Satisfiability)

Moves X_i	Booleans $\mathbb{B}$
Outcomes R	Boolean $\mathbb{B}$
Goal quantifier ϕ_i	Existential quantifier $\exists\colon K_{\mathbb{B}}\mathbb{B}$
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Optimal strategy

$$\text{next}_k(x_0, \ldots, x_{k-1}) \quad = \quad x_k \text{ such that } \exists x_{k+1} \ldots \exists x_{n-1} q(\vec{x})$$

(if possible)

Theorem (Main Theorem for Finite Games)

If ϕ_k are attainable with selection functions ε_k then

$$\mathsf{next}_k(x_0, \ldots, x_{k-1}) \stackrel{X_k}{=} \left(\left(\bigotimes_{i=k}^{n-1} \varepsilon_i \right) (q_{x_0,\ldots,x_{k-1}}) \right)_0$$

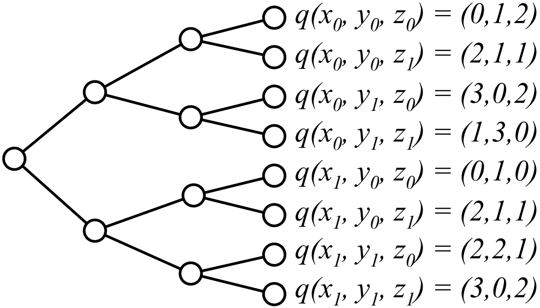
is an **optimal strategy** *for the game* $(R, (X_i)_{i<n}, (\phi_i)_{i<n}, q)$. *Moreover,*

$$\vec{a} = \left(\bigotimes_{i=0}^{n-1} \varepsilon_i \right) (q)$$

is the **strategic play**.

Nash equilibrium (sequential games)

$$q \colon X \times Y \times Z \to \mathbb{R}^3$$



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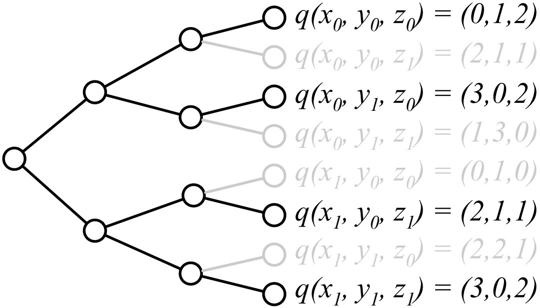

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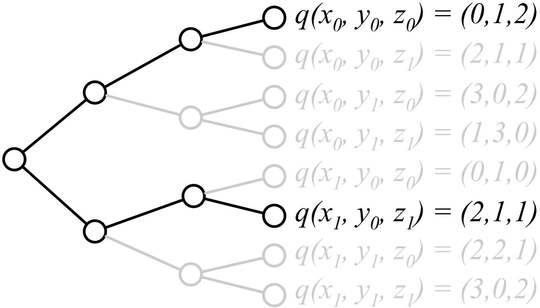

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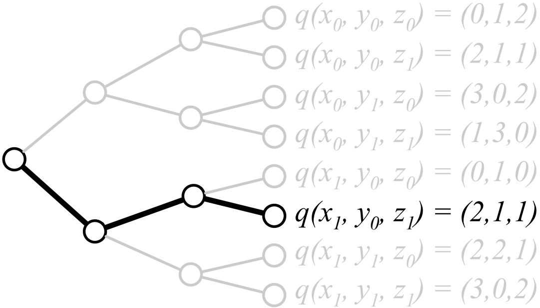

Backward Induction

Let $q \colon \Pi_{i=1}^{n} X_i \to \mathbb{R}^n$ be a payoff function

$\mathrm{argmax}_i(p)$ { $\qquad$ $[\mathrm{argmax}_i \colon (X_i \to \mathbb{R}^n) \to X_i]$

  return $x \in X_i$ such that $p(x)$ has maximal i-coordinate

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$\mathsf{sol}_i(x_1, \ldots, x_{i-1})$ { $\quad$ $[\mathsf{sol}_i \colon \Pi_{k=1}^{i-1} X_k \to \Pi_{k=i}^{n} X_k]$
  if $i = n + 1$ return $\langle\,\rangle$
  else
    $y := \mathrm{argmax}_i(\lambda x.q(\mathsf{sol}_{i+1}(x_1, \ldots, x_{i-1}, x)))$
    return $y * \mathsf{sol}_{i+1}(x_1, \ldots, x_{i-1}, y)$
}

Backward Induction

Let $q \colon \Pi_{i=1}^{n} X_i \to \mathbb{R}^n$ be a payoff function

$\mathrm{argmax}_i(p)$ { $\qquad$ $[\mathrm{argmax}_i \colon (X_i \to \mathbb{R}^n) \to X_i]$
  return $x \in X_i$ such that $p(x)$ has maximal i-coordinate
}

$\mathsf{sol}_i(x_1, \ldots, x_{i-1})$ { $\qquad$ $[\mathsf{sol}_i \colon \Pi_{k=1}^{i-1} X_k \to \Pi_{k=i}^{n} X_k]$
  if $i = n + 1$ return $\langle\,\rangle$
  else
    $y := \mathrm{argmax}_i(\lambda x . q(\mathsf{sol}_{i+1}(x_1, \ldots, x_{i-1}, x)))$
    return $y * \mathsf{sol}_{i+1}(x_1, \ldots, x_{i-1}, y)$
}

$\langle x_1, \ldots, x_n \rangle := \mathsf{sol}_1(\,)$

Backward Induction

Payoff function $q \colon \Pi_{i=1}^{n} X_i \to \mathbb{R}^n$

Each selection function

$$\mathrm{argmax}_i \colon (X_i \to \mathbb{R}^n) \to X_i$$

finds a point where the argument is i-maximal

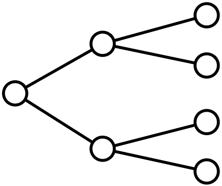
Product

$$\mathsf{sol}_1(\,) = \left(\bigotimes_{i=1}^{n} \mathrm{argmax}_i \right)(q)$$

calculates a **strategy profile in Nash equilibrium**.

Backtracking

good: $X \times Y \to \mathbb{B}$



Generic algorithm has type $(X \times Y \to \mathbb{B}) \to X \times Y$.

Backtracking

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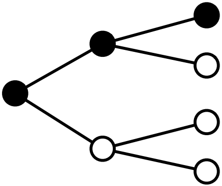

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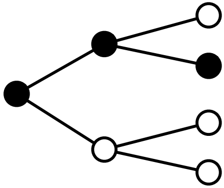

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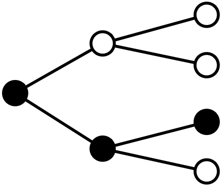

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For Instance – Eight Queens Problem

$\varepsilon(p)$ {
  for $(i := 1; i \leq 8; i{+}{+})$ do
    if $p(i)$ return i
  return 1
}

$[\, \varepsilon \colon (8 \to \mathbb{B}) \to 8 \,]$

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$$
\begin{array}{l}
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$$

$$
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\mathsf{sol}_i(x_1, \ldots, x_{i-1})\ \{ \\
\quad \text{if } i > 8 \text{ return } \langle\ \rangle \\
\quad \text{else} \\
\qquad y := \varepsilon(\lambda x_i.\mathsf{good}(\mathsf{sol}_{i+1}(x_1, \ldots, x_i))) \\
\qquad \text{return } y * \mathsf{sol}_{i+1}(x_1, \ldots, x_{i-1}, y) \\
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good: $8^8 \to \mathbb{B}$ checks if argument is solution to 8QP.

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finds argument $\varepsilon_i p \in 8$ such that $p(\varepsilon_i p)$ holds

$$\mathsf{sol}_1(\,) = \left(\bigotimes_{i=1}^{8} \varepsilon_i\right)(\mathsf{good})$$

calculates a solution to 8 queen problem.

Classical Arithmetic

Finite product interprets **bounded collection**

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E.g. consider the infinite PHP

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Equivalent (dialectica) to

$$\forall c^{\mathbb{N}\to n}, \forall \varepsilon^{J_{\mathbb{N}}\mathbb{N}} \exists b^n, p^{\mathbb{N}\to\mathbb{N}} (\overline{\varepsilon}_b p \geq \varepsilon_b p \wedge c(\overline{\varepsilon}_b p) = b)$$

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Witnessed by

$$b = c(\overline{\left(\bigotimes_{i=0}^{n-1} \varepsilon_i\right)(\max)})$$

Outline

Finite but Unbounded Games

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Example (Chess)

Moves X_i	Valid chess moves
Outcomes R	White, black, draw, e.g. $\{-1, 0, 1\}$
Goal quantifier ϕ_{2i}	Maximisation function
Goal quantifier ϕ_{2i+1}	Minimisation function
Outcome function q	Adjudication on a given play α

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The game is drawn, upon a correct claim by the player having the move, if

a. *he writes on his scoresheet, and declares to the arbiter his intention to make a move which shall result in the last* 50 *moves having been made by each player without the movement of any pawn and without the capture of any piece*, or
b. *the last* 50 *consecutive moves have been made by each player without the movement of any pawn and without the capture of any piece.*

With this rule, it can be shown that the game is finite, assuming that given the option to call for a draw, at least one player will do so.

Finite but Unbounded Games

Definition (Tuple $(R, (X_i)_{i \in \mathbb{N}}, (\phi_i)_{i \in \mathbb{N}}, q)$ where...)

- R is the set of possible **discrete** outcomes
- X_i is the set of available moves X_i at round $i \in \mathbb{N}$
- $\phi_i \colon K_R X_i$ are goal quantifiers for round $i \in \mathbb{N}$
- $q \colon \Pi_{i=0}^{\infty} X_i \to R$ is a **continous** outcome function

Optimal Strategies

Definition (Strategic Play)

Given strategy next_k and partial play $\vec{a} = a_0, \ldots, a_{k-1}$, the **strategic extension** of $\vec{a}$ is $\beta^{\vec{a}} = \beta^{\vec{a}}(k), \beta^{\vec{a}}(k+1), \ldots$ where

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Definition (Optimal Strategy)

Strategy next_k is **optimal** if for any partial play $\vec{a}$

$$q(\vec{a} * \beta^{\vec{a}}) = \phi_k(\lambda x_k.q(\vec{a} * x_k * \beta^{\vec{a},x_k})).$$

Finite but Unbounded Games

Theorem (Main Theorem for Finite but Unbounded Games)

If ϕ_k are attainable with selection functions ε_k then

$$\mathsf{next}_k(x_0, \ldots, x_{k-1}) \stackrel{X_k}{=} \left(\left(\bigotimes_{i=k}^{\infty} \varepsilon_i \right) (q_{x_0, \ldots, x_{k-1}}) \right)_0$$

is an **optimal strategy** *for the game $(R, (X_i)_{i \in \mathbb{N}}, (\phi_i)_{i \in \mathbb{N}}, q)$. Moreover,*

$$\alpha = \left(\bigotimes_{i=0}^{\infty} \varepsilon_i \right) (q)$$

is the **strategic play**.

Classical Analysis

Mathematical analysis is based on **comprehension**

$$\exists f^{\mathbb{N}\to\mathbb{B}} \forall n^{\mathbb{N}} (fn \leftrightarrow A_n).$$

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Countable choice is classically computational up to $\mathbf{DNS}$

$$\forall n^{\mathbb{N}} \neg\neg A_n \to \neg\neg \forall n^{\mathbb{N}} A_n.$$

Double negation shift

The double negation shift $\mathbf{DNS}$

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corresponds to the type

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The type of the **countable product** of selection functions!

Bar recursion

Not a coincidence!

Modified bar recursion is equivalent to

$$\bigotimes_{i=k}^{\infty} \varepsilon_i \overset{J\Pi X_i}{=} \varepsilon_k \otimes \left(\bigotimes_{i=k+1}^{\infty} \varepsilon_i \right)$$

Spector's bar recursion is equivalent to

$$\left(\bigotimes_{i=k}^{\infty} \varepsilon_i \right)(q) \overset{\Pi X_i}{=} \begin{cases} \mathbf{c} & \text{if } k < l(q(\mathbf{c})) \\ \left(\varepsilon_k \otimes \left(\bigotimes_{i=k+1}^{\infty} \varepsilon_i \right) \right)(q) & \text{otherwise} \end{cases}$$

Summary and Further Applications

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- E.g. Nash equilibrium, backtracking, Bekič's lemma

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- Functional interpretations (**proof mining**)

Theorems	$\mapsto$	games
Proofs	$\mapsto$	winning strategies

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5. Other places where $\bigotimes$ appear?

References

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