Instances of Bar Recursion as Products of Selection Functions

Paulo Oliva

Queen Mary, University of London, UK

(joint work with Martín Escardó)

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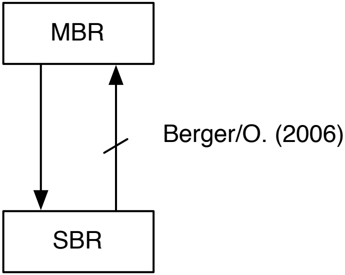
- Berardi et al. (1999) and Berger/O. (2005)

$$\text{Analysis} \overset{\text{(realizability)}}{\Rightarrow} \text{T} + \text{modified BR}$$

Bar Recursion – Overview



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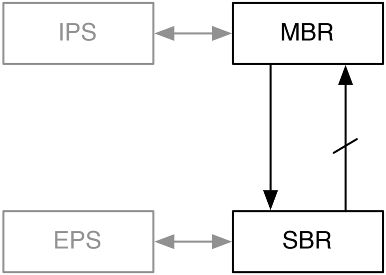

Outline

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$\varepsilon\colon JX$ is a **selection function for** $\phi\colon KX$ if

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Remark: Not all quantifiers have a selection function

Products

Consider two products

$$\begin{aligned} \otimes_{\mathsf{q}} \quad &: \quad KX \times KY \;\rightarrow\; K(X \times Y) \\ \otimes_{\mathsf{s}} \quad &: \quad JX \times JY \;\rightarrow\; J(X \times Y) \end{aligned}$$

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and

$$(\varepsilon \otimes_{\mathsf{s}} \delta)(p^{X \times Y \to R}) \stackrel{X \times Y}{:=} (a, b(a))$$

where $a := \varepsilon(\lambda x^X . p(x, b(x)))$ and $b(x) := \delta(\lambda y^Y . p(x, y))$

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Given sel. fct. $\varepsilon \colon JX$ we define a quantifier $\overline{\varepsilon} \colon KX$ as

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Theorem

$\overline{\varepsilon \otimes_{\mathsf{s}} \delta} = \overline{\varepsilon} \otimes_{\mathsf{q}} \overline{\delta}$

Product in Practice

Quantifier	**Sel. fct.**	
fix	fix	Bekič's lemma
sup	argsup	Backward induction
$\exists$	ε term	Epsilon method
$\exists$	search	Backtracking
$\overline{\varepsilon}$	ε	Bar recursion

Product in Practice

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In general, product computes optimal strategies and outcome

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1. Selection Functions
2. Modified Bar Recursion
3. Spector's Bar Recursion

Interpreting Classical Analysis

$$\mathsf{PA}^2 + \mathsf{CA}$$

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$$\underbrace{\mathsf{HA}^\omega + \mathsf{AC}_0}_{\text{semi-int.}} + \mathsf{DNS}$$

Double Negation Shift

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The type of the **countable product** of selection functions!

Implicitly Controlled Product

Given a family of selection functions $\{\varepsilon_i\}_{i \in \mathbb{N}}$, let

$$\mathsf{IPS}_i(\varepsilon) = \varepsilon_i \otimes_{\mathsf{s}} (\mathsf{IPS}_{i+1}(\varepsilon))$$

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$$\mathsf{IPS} \quad \mathsf{mr} \quad \forall n^{\mathbb{N}} JA(n) \to J \forall n^{\mathbb{N}} A(n)$$

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IPS *is primitive recursively equivalent to* MBR

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Interpreting Analysis via Dialectica Interpretation

Spector reduced interpretation of DNS to the following:

Spector's Equation

Given $\omega, \varepsilon_{(\cdot)}$ and q find $n, \alpha, p_{(\cdot)}$ satisfying

$$\begin{array}{lcl} n & \stackrel{\mathbb{N}}{=} & \omega(\alpha) \\ \alpha(n) & \stackrel{A_n}{=} & \varepsilon_n(p_n) \\ p_n(\alpha(n)) & \stackrel{R}{=} & q(\alpha) \end{array}$$

Explicitly Controlled Product

A solution to these equations can be computed with

$$\mathsf{EPS}_s(\varepsilon) = \begin{cases} \lambda q.\mathbf{0} & \text{if } \omega_s(\mathbf{0}) < |s| \\ \varepsilon_i \otimes_{\mathsf{s}} \lambda x.(\mathsf{EPS}_{s*x}(\varepsilon)) & \text{otherwise} \end{cases}$$

Product of sel. fcts. with *explicty control* ω on termination

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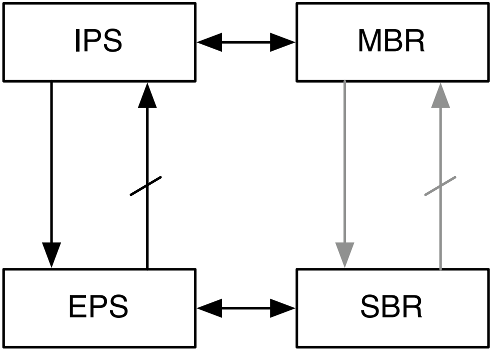
Theorem

EPS *is primitive recursively equivalent to* SBR

Summary



For Details See:

M. Escardó and P. Oliva
Selection functions, bar recursion and backward induction
Mathematical Structures in Computer Science, to appear

M. Escardó and P. Oliva
Instances of bar recursion as iterated products of selection functions and quantifiers
Computability in Europe CiE'2010