Selection Functions in Proof Theory

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(joint work with Martín Escardó)

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$$(X \to \mathbb{B}) \to \mathbb{B}$$

$$\exists^X, \forall^X \quad : \quad (X \to \mathbb{B}) \to \mathbb{B}$$

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\begin{aligned}
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KX \quad &\equiv \quad (X \to R) \to R
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(i) Both are **strong monads** ($T \in \{K, J\}$)

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X \to TX \qquad T^2 X \to TX \qquad (X \wedge TY) \to T(X \wedge Y)
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(ii) There is a **monad morphism**

$$JX \to KX$$

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(ii) There is a **monad morphism**

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(iii) And, if $R \to X$ then

$$KX \to JX$$

Let us call elements $\phi\colon KX$ **generalised quantifiers**

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$\phi\colon KX$ is called **attainable** if it has a selection function

Strength

Strength gives (for $T \in \{K, J\}$)

$$\otimes \quad : \quad TX \times TY \to T(X \times Y)$$

Products of selection functions and generalised quantifiers

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Products of selection functions and generalised quantifiers

Moreover, given $(\overline{\cdot}) \colon JX \to KX$ we have

$$\overline{\varepsilon} \otimes \overline{\delta} = \overline{\varepsilon \otimes \delta}$$

The product of attainable quantifiers is attainable

Advert

Selection functions and in particular their product $\otimes : JX \times JY \to J(X \times Y)$ arise in

- **Game theory** (*Backward induction*)
- **Fixed point theory** (*Bekič's lemma*)
- **Algorithms** (*Backtracking*)
- **Proof theory** (*as we shall see ...*)

Outline

1. Known
   - Hilbert's epsilon terms
   - Kreisel's no-counterexample functionals
2. Novel
   - New negative translation
   - Alternative interpretation of analysis
3. Conclusions

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Hilbert: Problem and Approach to Solution

Problem

Prove the consistency of mathematics by finitary means

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Approach

1. Eliminate quantification via "epsilon" terms

$$\exists^X p \to p(\varepsilon_p) \qquad \text{(epsilon axioms)}$$

2. Show that Maths can be done in the "epsilon" calculus

3. Show that any finite set of axioms has a model

Hilbert's Heritage

Let's have a closer look at the epsilon terms (and axiom)

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A **selection functions** for $\exists^X : KX$!

No-counterexample Interpretation

Kreisel's **main observation**:

$$\mathsf{PA} \vdash \exists x^X \forall y^Y p(x, y)$$

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$$\mathsf{PA} \vdash \exists x^X \forall y^Y p(x, y) \quad \Rightarrow \quad \forall f^{X \to Y} p(\varepsilon f, f(\varepsilon f))$$

for a **recursive** $\varepsilon \colon (X \to Y) \to X$.

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for a **recursive** $\varepsilon \colon (X \to Y) \to X$.

Although **witness** x^X may not be produced recursively, the **selection function** ε is!

Classical logic is interpreted by moving from

elements of X $\quad\mapsto\quad$ selection functions over X

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Gödel-Gentzen Negative Translation

Let $R = \bot$. Then $KX = \neg\neg X$.

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\begin{aligned}
P^K &= KP \\
(A \wedge B)^K &= A^K \wedge B^K \\
(A \vee B)^K &= K(A^K \vee B^K) \\
(A \to B)^K &= A^K \to B^K \\
(\forall x A)^K &= \forall x A^K \\
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What if we use J instead of K?

The Peirce Translation

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\begin{aligned}
P^J &= JP \\
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As $KA^K \to A^K$ we also have $JA^J \to A^J$

The Peirce Translation

Theorem

A is provable in minimal logic plus Peirce's law

$$\underbrace{((A \to R) \to A)}_{JA} \to A$$

if and only if A^J is provable in minimal logic

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Theorem

A is provable in minimal logic plus Peirce's law

$$\underbrace{((A \to R) \to A)}_{JA} \to A$$

if and only if A^J is provable in minimal logic

In fact, this is more fundamental

Gödel-Gentzen follows from Peirce (since $J \mapsto K$)

Interpreting Mathematical Analysis

Mathematical analysis is based on **comprehension**

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$$\forall n^{\mathbb{N}} \exists b^{\mathbb{B}} A(n, b) \to \exists f \forall n A(n, fn)$$

Countable choice is classically computational up to **DNS**

$$\forall n \neg\neg A(n) \to \neg\neg \forall n A(n)$$

Double Negation Shift

The double negation shift **DNS**

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corresponds to the type

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DNS is interpreted by using that $\perp \to A_n$ and reducing to

$$\Pi_n J A_n \to J \Pi_n A_n$$

The type of the **countable product** of selection functions!

Bar Recursion

Recall

$$\otimes \quad : \quad JX \times JY \to J(X \times Y)$$

Let

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Let

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Then

$$\bigotimes_0 \quad \text{realizes} \quad \Pi_n J A_n \to J \Pi_n A_n$$

and hence (with Gödel's T) realises **full classical analysis**

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2. Intuition

Predicate Logic	Fixed use of EM	$\varepsilon \otimes \delta$
Arithmetic	Unbounded use of EM	$\bigotimes_{i=0}^{n} \varepsilon_i$
Analysis	Countable use of EM	$\bigotimes_{i=0}^{\infty} \varepsilon_i$

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Analyse concrete proofs in mathematics from such perspective

For more information

M. Escardó and P. Oliva
Selection functions, bar recursion and backward induction
To appear in MSCS, 42 pages, 2010

M. Escardó and P. Oliva
Computational interpretations of analysis via products of selection functions
Proceedings of CiE, 10 pages, 2010

M. Escardó and P. Oliva
The Peirce translation and the double negation shift
Submitted for publication, 10 pages, Jan 2010