Spector's Bar Recursion as a Product of Selection Functions

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Outline

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Generalised quantifiers

$$\phi : (X \to R) \to R$$

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For instance

Operation	ϕ	:	$(X \to R) \to R$
Quantifiers	$\forall_X, \exists_X$	:	$(X \to \mathbb{B}) \to \mathbb{B}$
Integration	$\int_0^1$	:	$([0,1] \to \mathbb{R}) \to \mathbb{R}$
Supremum	$\sup_{[0,1]}$	:	$([0,1] \to \mathbb{R}) \to \mathbb{R}$
Limit	$\lim$	:	$(\mathbb{N} \to R) \to R$
Fixed point operator	fix_X	:	$(X \to X) \to X$

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$$\phi : (X \to R) \to R \qquad (\equiv K_R X)$$

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Definition (Product of Generalised Quantifiers)

Given $\phi \colon K_R X$ and $\psi \colon K_R Y$ define $\phi \otimes \psi \ : \ K_R(X \times Y)$

$$(\phi \otimes \psi)(p) \stackrel{R}{:\equiv} \phi(\lambda x^X . \psi(\lambda y^Y . p(x,y)))$$

where $p \colon X \times Y \to R$.

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$$(\exists_X \otimes \forall_Y)(p^{X \times Y \to \mathbb{B}}) \quad \stackrel{\mathbb{B}}{\equiv} \quad \exists x^X \forall y^Y p(x,y)$$

$$(\sup \otimes \textstyle\int)(p^{[0,1]^2 \to \mathbb{R}}) \quad \stackrel{\mathbb{R}}{\equiv} \quad \sup_x \int_0^1 p(x,y) dy$$

Theorem (Mean Value Theorem)

For any $p \in C[0,1]$ *there is a point* $a \in [0,1]$ *such that*

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Theorem (Witness Theorem)

For any $p \colon X \to \mathbb{B}$ *there is a point* $a \in X$ *such that*

$$\exists x^X p(x) \quad \Leftrightarrow \quad p(a)$$

(similar to Hilbert's ε*-term).*

Let $J_R X :\equiv (X \to R) \to X$.

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holds for all $p \colon X \to R$.

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Definition (Attainable Quantifiers)

A generalised quantifier $\phi\colon K_R X$ is called **attainable** if it has a selection function $\varepsilon\colon J_R X$.

The Mapping $(\overline{\cdot}) : J_R \mapsto K_R$

Every element

$$\varepsilon \quad : \quad J_R X$$

is a selection function for the (attainable) quantifier

$$\overline{\varepsilon} p \overset{R}{:=} p(\varepsilon p).$$

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We define a product of selection functions such that

$$\overline{\varepsilon \otimes \delta} = \overline{\varepsilon} \otimes \overline{\delta}$$

Definition (Product of Selection Functions)

Given $\varepsilon \colon J_R X$ and $\delta \colon J_R Y$ define $\varepsilon \otimes \delta \colon J_R(X \times Y)$ as

$$(\varepsilon \otimes \delta)(p^{X \times Y \to R}) \overset{X \times Y}{:=} (a, b(a))$$

where

$$\begin{array}{lcl} a & := & \varepsilon(\lambda x . p(x, b(x))) \\ b(x) & := & \delta(\lambda y . p(x, y)). \end{array}$$

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In fact, $(\varepsilon \otimes \delta)(p) = (a, b(a))$.

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Proof.

Let $a = \varepsilon(\lambda x.q(x, b(x)))$ and $b(x) = \delta(\lambda y.q(x, y))$. Then

$$\begin{aligned}(\overline{\varepsilon} \otimes \overline{\delta})(q) &= \overline{\varepsilon}(\lambda x.\overline{\delta}(\lambda y.q(x, y))) \\ &= \overline{\varepsilon}(\lambda x.q(x, b(x))) \\ &= q(a, b(a)) \\ &= q((\varepsilon \otimes \delta)(q)) \\ &= (\overline{\varepsilon \otimes \delta})(q).\end{aligned}$$

□

Outline

Finite Games (n rounds)

X_i	available **moves** at round i
R	set of **possible outcomes**
$q\colon \Pi_{i=0}^{n-1} X_i \to R$	**outcome function**
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Definition (Optimal outcome and moves)

For $\vec{x} \equiv x_0, \ldots, x_{k-1}$ call

$$w_{\vec{x}} := \bigotimes_{i=k}^{n-1} (\phi_i)(q_{\vec{x}})$$

the **optimal outcome** of sub-game $x_0, \ldots, x_{k-1}$.

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Move a_k is an **optimal move** at round k if $w_{\vec{x}} = w_{\vec{x} * a_k}$.

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Theorem

If ϕ_k are attainable (with selection functions ε_k) then

(i) $\vec{a}_{\vec{x}} := \bigotimes_{i=k}^{n-1}(\varepsilon_i)(q)$
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(iii) *Let* $p_k := \lambda x_k . w_{a_0, \ldots, a_{k-1} * x_k}$. *Then*

$$a_k = \varepsilon_k(p_k) \quad \text{and} \quad p_k(a_k) = p_j(a_j) = q(\vec{a}).$$

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(iii) *Let* $p_k := \lambda x_k . w_{\overline{\alpha}k * x_k}$. *We have, for all* k,

$$\alpha(k) = \varepsilon_k(p_k) \quad \text{and} \quad p_k(\alpha(k)) = q(\alpha)$$

In Other Words...

Theorem

Given

$$\begin{array}{rcl} \varepsilon_k & : & (X_k \to R) \to X_k \\ q & : & \Pi_{i=0}^{\infty} X_i \to R \end{array}$$

we have, for all n,

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Except that infinite products might not be defined!

R might not be discrete.

The Good News

Spector's Problem

Given $\omega, \varepsilon_{(\cdot)}$ and q find $n, \alpha, p_{(\cdot)}$ satisfying

$$\begin{array}{lcl} n & \overset{\mathbb{N}}{=} & \omega(\alpha) \\ \alpha(n) & \overset{X_n}{=} & \varepsilon_n(p_n) \\ p_n(\alpha(n)) & \overset{R}{=} & q(\alpha) \end{array}$$

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Good News. We don't need to play optimally forever.

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Conditional iteration

Iterated product

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Spector's solution

$$\bigotimes_s(\varepsilon)(q) \overset{\Pi_{i=|s|}^{\infty} X_i}{=} \begin{cases} \mathbf{0} & \text{if } \omega_s(\mathbf{0}) < |s| \\ (\varepsilon_{|s|} \otimes \lambda x. \bigotimes_{s*x}(\varepsilon))(q) & \text{otherwise.} \end{cases}$$

Spector's Oversight (?)

In finding the solution (a product of **selection functions**)

$$\bigotimes_s(\varepsilon)(q) \stackrel{\Pi_{i=k}^{\infty} X_i}{=\!=} \begin{cases} \mathbf{0} & \text{if } \omega_s(\mathbf{0}) < |s| \\ (\varepsilon_{|s|} \otimes \lambda x. \bigotimes_{s*x}(\varepsilon))(q) & \text{otherwise.} \end{cases}$$

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Spector generalises recusion scheme as product of **quantifiers**!

$$\bigotimes_s(\phi)(q) \stackrel{R}{=} \begin{cases} g(\mathbf{0}) & \text{if } \omega_s(\mathbf{0}) < |s| \\ (\phi_{|s|} \otimes \lambda x . \bigotimes_{s*x}(\phi))(q) & \text{otherwise.} \end{cases}$$

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$$\bigotimes_s(\phi)(q) \overset{R}{=} \begin{cases} g(\mathbf{0}) & \text{if } \omega_s(\mathbf{0}) < |s| \\ \phi_{|s|}(\lambda x. \bigotimes_{s*x}(\phi)(q_x)) & \text{otherwise.} \end{cases}$$

For more information

M. Escardo and P. Oliva
Selection functions, bar recursion and backward induction
Submitted, July 2009

M. Escardo and P. Oliva
Instances of bar recursion as iterated products of selection functions and quantifiers
In preparation