Functional Interpretations

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Queen Mary, University of London

Coimbra, 9 September 2009

Outline

Outline

1. Functional Interpretations
2. Linear Logic
3. Unifying Interpretation
4. Recovering Usual Interpretations

Functionals

$$
\begin{array}{lcl}
1, 2, \ldots & : & \mathbb{N} \\
\mathsf{succ} & : & \mathbb{N} \to \mathbb{N} \\
\mathsf{min} & : & (\mathbb{N} \to \mathbb{N}) \to \mathbb{N} \\
\mathsf{It}_0 & : & (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N}) \\
\ldots & & \ldots
\end{array}
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Functionals of finite types

$$\rho, \tau \quad :\equiv \quad \mathbb{N} \quad | \quad \rho \to \tau$$

Functional interpretations

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$\#\mathbb{N}$ is finite	$\mathbb{N}$	$\forall n^{\mathbb{N}}(n \leq f)$

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$\#\mathbb{N}$ is finite	$\mathbb{N}$	$\forall n^{\mathbb{N}}(n \leq f)$
$\sqrt{2}$ is irrational	$\mathbb{Q} \to \mathbb{N}$	$\forall q^{\mathbb{Q}}(\lvert\sqrt{2} - q\rvert \geq 1/2^{fq})$

Functional interpretations

Formula $A \quad \sim \quad$ **Set of functionals** $[A] \subseteq \rho_A$

Provable $A \quad \sim \quad [A] \neq \emptyset$

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Effective Soundness. Algorithms ϕ, ψ exist such that

$$\overset{\pi}{\vdash} A \quad \Rightarrow \quad \overset{\psi\pi}{\vdash} \phi\pi \in [A]$$

$\phi\pi$ a "realizer" for A

$\psi\pi$ verification proof that $\phi\pi$ is a "realizer"

Applications

$$\mathsf{T} \overset{\pi}{\vdash} A \quad \Rightarrow \quad \mathsf{S} \overset{\psi\pi}{\vdash} [A](\phi\pi)$$

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- **Unprovability**. $\mathsf{T} \nvdash A$ $\quad ([A](t) \to \perp)$
- **Models**. $\mathcal{M}_{\mathsf{T}} \models A$ iff $\mathcal{M}_{\mathsf{S}} \models \exists x [A](x)$

Functional interpretations

Functional interpretation	
BHK interpretation	BHK'30's
Number realizability	Kleene'45
Q-variant of realizability	Kleene'45
Dialectica interpretation	Gödel'58
Modified realizability	Kreisel'62
Realizability with truth	Aczel'68
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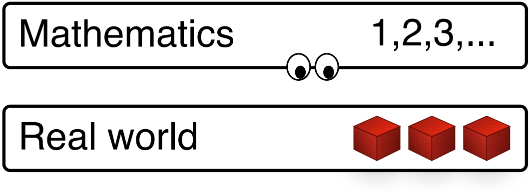
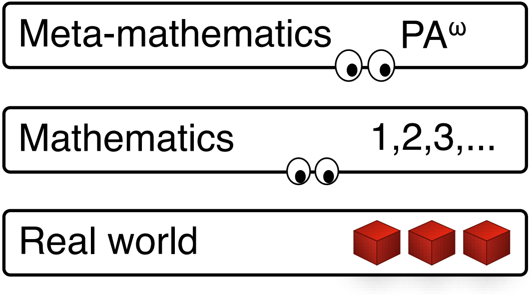
Bounded functional interpretations in Gilda Ferreira's talk


Mathematics
1,2,3,...
Real world


Meta-mathematics
PAω
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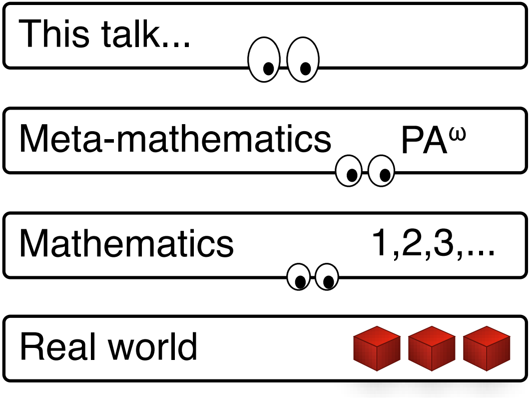
This talk...
Meta-mathematics
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Some questions

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- How are they different?

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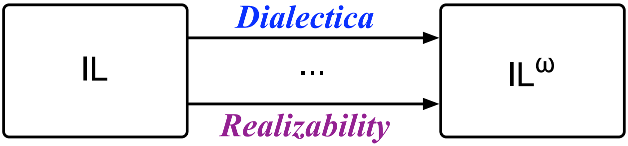
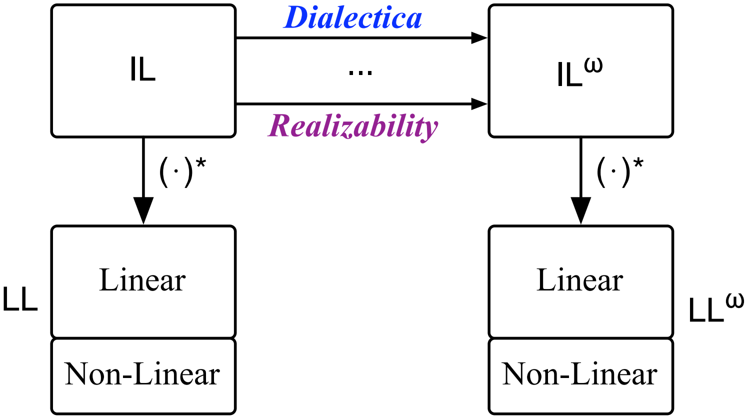
- What do they have in common?
- How are they different?
- Is there a general recipe?
- How many more can we expect to discover?


IL
Dialectica
...
Realizability
IL^ω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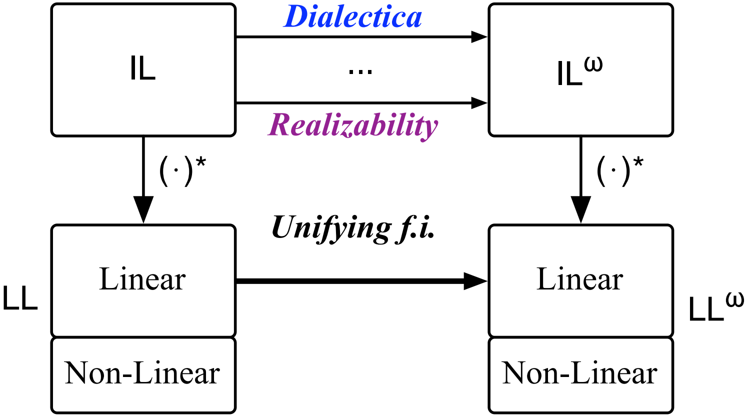
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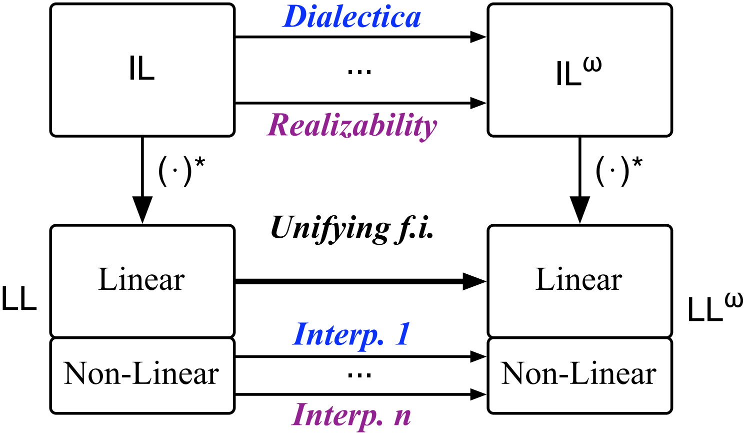
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Outline

Linear logic

Linear logic = keep track of **contractions**

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}$$

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This can be viewed as two steps

$$\cfrac{\cfrac{\Gamma, A, A \vdash B}{\Gamma, !A, !A \vdash B}\ (\text{dereliction})}{\Gamma, !A \vdash B}\ (\text{contraction})$$

Linear logic

Additive conjunction

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}$$

Multiplicative conjunction

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$

Linear logic

Linear (pure) part	**Non-linear part**
$A \otimes B$	$!A$
$A \multimap B$	
$A \& B$	
$A \oplus B$	
$\exists x A$	
$\forall x A$	

Embedding IL into ILL

Girard's translation

$$
\begin{aligned}
(A \wedge B)^* &:\equiv A^* \,\&\, B^* \\
(A \vee B)^* &:\equiv\, !A^* \oplus !B^* \\
(A \rightarrow B)^* &:\equiv\, !A^* \multimap B^* \\
(\forall x A)^* &:\equiv \forall x A^* \\
(\exists x A)^* &:\equiv \exists x !A^*
\end{aligned}
$$

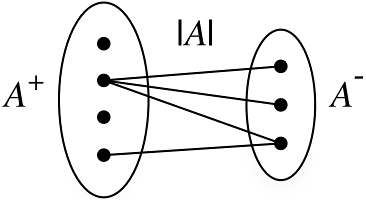
Lemma

IL $\vdash A$ *if and only if* ILL $\vdash A^*$.

Outline

Interpretation

Interpret formulas as **bipartite graphs** $|A|^x_y \subseteq A^+ \times A^-$



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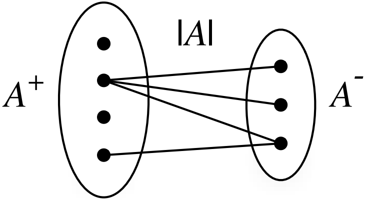

Then $[A] :\equiv \{x^{A^+} \ : \ \forall y^{A^-} |A|^x_y\}$.

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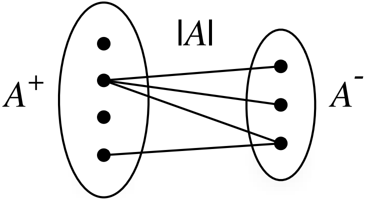

Then $[A] :\equiv \{x^{A^+} \; : \; \forall y^{A^-} |A|^x_y\}$.

Ps. Think of one-move two-player games (Eloise vs. Abelard).

Sum of games $A \oplus B$

Eloise chooses whether to play A or B

$$|A \oplus B|^{\mathsf{inj}_i x}_{\langle y, w\rangle} \quad :\equiv \quad \begin{cases} |A|^x_y & \text{if } i = 0 \\ |B|^x_w & \text{if } i = 1 \end{cases}$$

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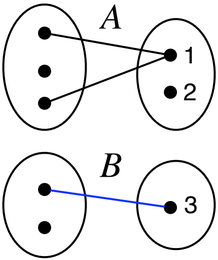

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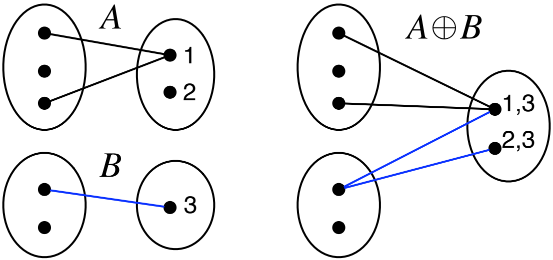

The games $A \mathbin{\&} B$

Abelard chooses whether to play A or B

$$|A \mathbin{\&} B|^{\langle x,v\rangle}_{\mathsf{inj}_i y} \quad :\equiv \quad \begin{cases} |A|^x_y & \text{if } i = 0 \\ |B|^v_y & \text{if } i = 1 \end{cases}$$

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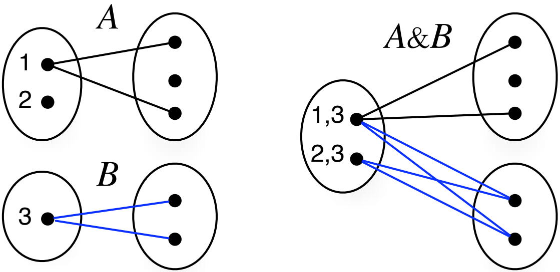

Product of games $A \otimes B$

Eloise must win both games A and B

$$|A \otimes B|^{\langle x,v\rangle}_{\langle y,w\rangle} \quad :\equiv \quad |A|^x_y \text{ and } |B|^v_w$$

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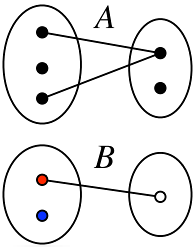

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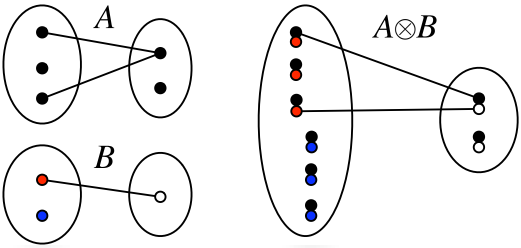

Relative games $A \multimap B$

Play game A relative to game B

$$|A \multimap B|^{\langle S,T \rangle}_{\langle x,w \rangle} \quad :\equiv \quad |A|^{x}_{S_{x,w}} \text{ implies } |B|^{T_x}_{w}$$

where

- $T \in \mathcal{B}_f(A^+, B^+)$
- $S \in \mathcal{B}_f(A^+ \times B^-, A^-)$.

$\mathcal{B}_f(X, Y) \equiv$ functional bipartite graphs between X and Y

Replicating games

Play several copies of game A in parallel

$$|!A|^x_* \quad :\equiv \quad \forall y^{A^-} |A|^x_y$$

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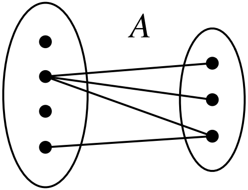

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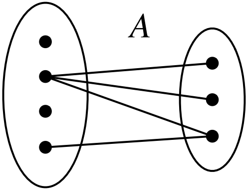

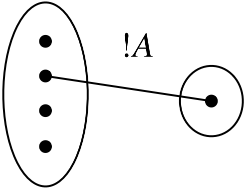

Soundness

A is provable in linear logic

$$\downarrow$$

bipartite graph $|A|^x_y$ has a covering point

(i.e. there exists an x^{A^+} such that $\forall y^{A^-} |A|^x_y$)

(or, Eloise has a winning move in game A)

Replicating games (revisited)

What if **infinite graphs** are allowed?

Play "several" copies of a game in parallel

$$|!A|_S^x \quad :\equiv \quad \forall y \in S \, |A|_y^x$$

where $(!A)^+ = A^+$ and $(!A)^- = \hat{\mathcal{P}}(A^-)$.

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Which subsets give rise to a sound interpretation?

Exponentials: Conditions

The kind of move-sets $\hat{\mathcal{P}}(A^-)$ need to satisfy:

There exists terms $\boldsymbol{\eta}, \boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ such that

(I) **Singleton**
Every element $\boldsymbol{x} \in A^-$ belongs to a set $\boldsymbol{\eta x} \in \hat{\mathcal{P}}(A^-)$

(II) **Finite union**
Sets $\boldsymbol{y}_i \in \hat{\mathcal{P}}(A^-)$ are contained in set $\boldsymbol{\epsilon y}_0 \boldsymbol{y}_1 \in \hat{\mathcal{P}}(A^-)$

(III) **Indexed union**
For $\boldsymbol{x} \in \boldsymbol{b}$, the set $\boldsymbol{hx} \in \hat{\mathcal{P}}(A^-)$ is in $\boldsymbol{\mu hb} \in \hat{\mathcal{P}}(A^-)$

Instances satisfying (I, II, III)

- **Whole set**

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- **Singleton sets** (*assuming decidability*)

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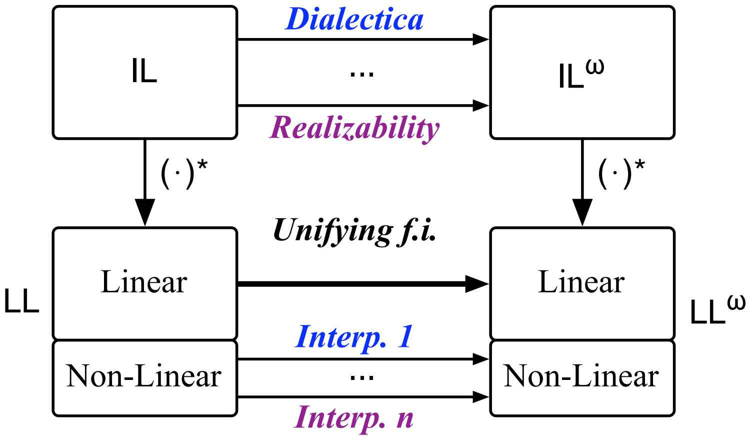
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Usual interpretations of IL

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*Joint work with Gilda Ferreira

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Kreisel mod. realizability

$|A^\circ|^{\boldsymbol{x}} \multimapboth (\boldsymbol{x} \text{ mr } A)^\circ$

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Kreisel mod. realizability

$|A^\circ|^{\boldsymbol{x}} \multimap\!\!\circ\ (\boldsymbol{x} \text{ mr } A)^\circ$

Stein interpretation

$|A^*|^{\boldsymbol{x}}_{\boldsymbol{y}} \multimap\!\!\circ\ (A_s(\boldsymbol{x}; \boldsymbol{y}))^*$

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Usual interpretations of IL

- **Whole set** — Kreisel mod. realizability

  $|!A|^{\boldsymbol{x}} :\equiv \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ — $|A^\circ|^{\boldsymbol{x}} \multimapboth (\boldsymbol{x} \text{ mr } A)^\circ$

- **Countable set** — Stein interpretation

  $|!A|^{\boldsymbol{x}}_{\boldsymbol{\alpha}} :\equiv \forall n |A|^{\boldsymbol{x}}_{\boldsymbol{\alpha}(n)}$ — $|A^*|^{\boldsymbol{x}}_{\boldsymbol{y}} \multimapboth (A_s(\boldsymbol{x}; \boldsymbol{y}))^*$

- **Finite sets** — Diller-Nahm inter.

  $|!A|^{\boldsymbol{x}}_{\boldsymbol{a}} :\equiv \forall \boldsymbol{y} \in \boldsymbol{a} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ — $|A^*|^{\boldsymbol{x}}_{\boldsymbol{y}} \multimapboth (A_{dn}(\boldsymbol{x}; \boldsymbol{y}))^*$

- **Singleton sets**

  $|!A|^{\boldsymbol{x}}_{\boldsymbol{y}} :\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$

*Joint work with Gilda Ferreira

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- **Singleton sets** — Gödel Dialectica inter.

  $|!A|^{\boldsymbol{x}}_{\boldsymbol{y}} :\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ — $|A^*|^{\boldsymbol{x}}_{\boldsymbol{y}} \multimapboth (A_D(\boldsymbol{x}; \boldsymbol{y}))^*$

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Functional interpretations

Interpretation	
Number realizability	Kleene'45
Q-variant of realizability	Kleene'45
Dialectica interpretation	Gödel'58
Modified realizability	Kreisel'62
Realizability with truth	Aczel'68
Variant of dialectica	Diller/Nahm'72
Parmetrised interpretation	Stein'79

Replicating games (re-revisited)

We can also check whether A itself is **true**:

$$|!A|^x_S \quad :\equiv \quad (\forall y \in S\, |A|^x_y) \wedge A$$

This also leads to sound interpretations.

*Joint work with Jaime Gaspar

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This also leads to sound interpretations.

$$\Gamma \vdash A \quad \Rightarrow \quad F(\Gamma) \vdash F(A).$$

*Joint work with Jaime Gaspar

Summary

Interpretation	Non-linear $\lvert !A \rvert$	Embedding
Number realizability	$\forall m \lvert A \rvert^n_m$	$(\cdot)^\circ$
Modified realizability	$\forall \boldsymbol{y} \lvert A \rvert^{\boldsymbol{x}}_{\boldsymbol{y}}$	$(\cdot)^\circ$
Stein	$\forall n \, \lvert A \rvert^{\boldsymbol{x}}_{\boldsymbol{\alpha}_n}$	$(\cdot)^*$
Diller-Nahm	$\forall \boldsymbol{y} \in \boldsymbol{a} \, \lvert A \rvert^{\boldsymbol{x}}_{\boldsymbol{y}}$	$(\cdot)^*$
Dialectica interpretation	$\lvert A \rvert^{\boldsymbol{x}}_{\boldsymbol{y}}$	$(\cdot)^*$
Q-variant of realizability	$\forall m \lvert A \rvert^n_m \otimes A$	$(\cdot)^*$
Realizability with truth	$\forall m \lvert A \rvert^n_m \otimes A$	$(\cdot)^\circ$

Conclusions

Related Work

- Valéria de Paiva's dialectica categories. Masaru Shirahata's extension.
- Chu spaces. Andreas Blass game interpretation of LL.
- Martin Hyland's proof theory in the abstract. Categorical view of functional interpretations. $!A$ as monads.

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Open Questions

- Odd interpretations: Goodman's realizability with forcing, Lifschtz realizability interpretation
- Relation between functional interpretations of linear logic and known "models"

References

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