

On Variants of Modified Bar Recursion

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(Joint work with Martín Escardó)

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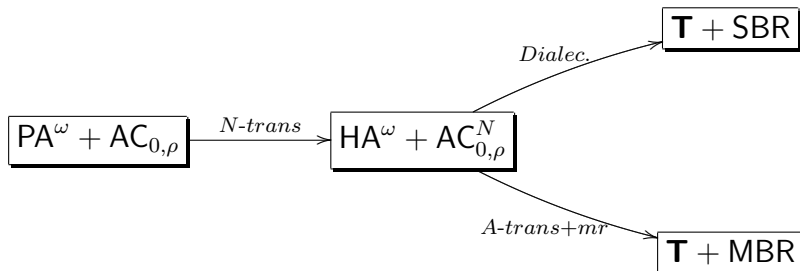
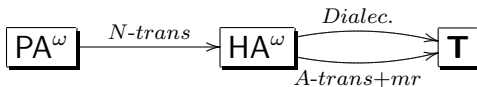
Outline

- 1 Background
- 2 Three Realizability Bar Recursions
 - BBC bar recursion
 - Berger's bar recursion
 - Escardo's bar recursion

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Interpretations of Arithmetic and Analysis



Primitive recursion vs Bar recursion

$$R(n) \stackrel{\tau}{=} \begin{cases} G & \text{if } n = 0 \\ H_n(R(n-1)) & \text{otherwise} \end{cases}$$

$$SBR(s^{\rho^*}) \stackrel{\tau}{=} \begin{cases} G_s & \text{if } Y(\hat{s}) < |s| \\ H_s(\lambda x^{\rho}.SBR(s * x)) & \text{otherwise.} \end{cases}$$

Primitive recursion vs Bar recursion

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The Challenge

$$\forall n \exists x A(n, x) \rightarrow \exists f \forall n A(n, fn)$$

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$$\forall n ((\exists x A^B(n, x) \rightarrow B) \rightarrow B) \wedge (\exists f \forall n A^B(n, fn) \rightarrow B) \rightarrow B$$

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Given realizers for

$$\forall n ((\exists x A^B(n, x) \rightarrow B) \rightarrow B)$$

$$\exists f \forall n A^B(n, fn) \rightarrow B$$

produce realizer for B .

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Given realizers for

$$\forall n ((A(n) \rightarrow B) \rightarrow A(n))$$

$$\forall n A(n) \rightarrow B$$

produce realizer for B .

The Challenge

Given

$$H_n : (A(n) \rightarrow B) \rightarrow A(n)$$

$$Y : \forall n A(n) \rightarrow B$$

Produce a realiser for B (or $\forall n A(n)$).

The Challenge

Given

$$H_n : (A(n) \rightarrow B) \rightarrow A(n)$$

$$Y : \forall n A(n) \rightarrow B$$

Produce a realiser for B (or $\forall n A(n)$).

Sketch of solution: (assume $s^{(\mathbb{N} \times \rho)^*} : \forall n \in s A(n)$)

$$\Psi(s)(n) \stackrel{\rho}{=} \begin{cases} s(n) & \text{if } n \in s \\ H_n(\lambda x^\rho. Y(\Psi(s * \langle n, x \rangle))) & \text{otherwise.} \end{cases}$$

Berardi, Bezem, Coquand (BBC) functional

$$\Psi(s) = s @ \lambda n. H_n(\lambda x. Y(\Psi(s * \langle n, x \rangle)))$$

- Efficient
- **Not** easy to prove total
- **Not** easy to prove it is a realiser

Berger's observation

Enough: Given

$$H_n : (A(k) \rightarrow B) \rightarrow A(n)$$

$$Y : \forall n A(n) \rightarrow B$$

Produce a realiser for $\forall n A(n)$.

Berger's observation

Enough: Given

$$H_n \quad : \quad (A(k) \rightarrow B) \rightarrow A(n)$$

$$Y \quad : \quad \forall n A(n) \rightarrow B$$

Produce a realiser for $\forall n A(n)$.

Sketch of solution: (assume $s^{\rho^*} : \forall n < |s| A(n)$)

$$\Psi(s)(n) \stackrel{\rho}{=} \begin{cases} s_n & \text{if } n < |s| \\ H_n(\lambda x^{\rho}. Y(\Psi(s * x))) & \text{otherwise.} \end{cases}$$

Berger's (MBR) functional

$$\Psi(s) = s @ \lambda n. H_n(\lambda x. Y(\Psi(s * \langle |s|, x \rangle)))$$

- **Not** very efficient
- Easy to prove total
(*by bar induction*)
- Easy to prove it is a realiser
(*by bar induction*)

Escardo's trick

Given

$$H_n : (A(n) \rightarrow B) \rightarrow A(n)$$

$$Y : \forall n A(n) \rightarrow B$$

Produce a realiser for $\forall n A(n)$.

Sketch of solution: (assume $s^{\rho^*} : \forall n < |s| A(n)$)

$$\Psi(s)(n) \stackrel{\rho}{=} \begin{cases} s_n & \text{if } n < |s| \\ H_n(\lambda x^{\rho}. Y(\Psi(s * \langle n, x \rangle))) & \text{otherwise.} \end{cases}$$

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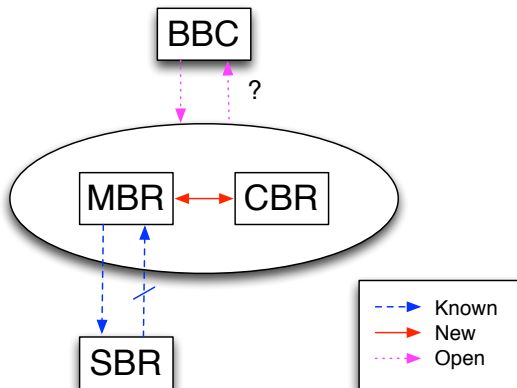
where $\dots \equiv \Psi(s)[|s|, n - 1]$.

Escardo's (CBR) bar recursion

$$\Psi(s) = s @ \lambda n. H_n(\lambda x. Y(\Psi(\overline{\Psi(s)}(n) * \langle n, x \rangle)))$$

- Efficient
- Easy to prove total
(by *course-of-value bar induction*)
- Easy to prove it is a realiser
(by *course-of-value bar induction*)

Main results



References

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