Recent developments around the Dialectica interpretation

Fernando Ferreira* & Paulo Oliva**

*Universidade de Lisboa & **Queen Mary, University of London

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Three Lectures

First Lecture

Introduction to the Dialectica and majorizability interpretations

Fernando Ferreira

Second Lecture

Injecting uniformities into classical mathematics

Fernando Ferreira

Third Lecture

Dialectica interpretation in the light of linear logic

Paulo Oliva

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Dialectica interpretation in the light of linear logic

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Outline

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1. Introduction
   - Contraction
   - Linear Logic
2. Dialectica Interpretation of LL
   - Interpretation of LL
   - Interpretation of IL
   - Characterisation
3. Uses of LL Interpretation
   - Unification
   - Hybrid Interpretations

Dialectica interpretation of IL

- Asymmetric

  $\exists x \forall y A_D(x; y)$ (*intuitionistically*)

  $\forall y \exists x A_D(x; y)$ (*classically*)

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Subtlety is in the interpretation of ! and ?

Importance of structural rules

- **Combinators**

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\begin{array}{lcl}
Kxy & \mapsto & x \\
Sxyz & \mapsto & xz(yz)
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How do they do it?

Move contractions from the conclusion to the premise

$$\vdash A \qquad \Rightarrow \qquad \neg A \vdash \bot$$

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Need to take contraction and weakening seriously

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Linear Logic!

Linear Logic (Girard 1987)

- Explicit treatment of **contraction**

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	conjunction	disjunction
additive	$\wedge$	$\vee$
multiplicative	$\otimes$	$⅋$

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	conjunction	disjunction
additive	$\wedge$	$\vee$
multiplicative	$\otimes$	$\invamp$

- **Refinement** of intuitionistic implication

$$A \rightarrow B \quad \equiv \quad !A \multimap B$$

Linear Logic with if-then-else

- Assume the boolean data type $\mathbb{B}$
- Instead of having $\wedge$ and $\vee$ add $\Diamond_b$
- Semantics:

  $A \Diamond_b B \equiv$ if b then A else B
- Define additive connectives as

  $A \wedge B \equiv \forall b^{\mathbb{B}} (A \Diamond_b B)$

  $A \vee B \equiv \exists b^{\mathbb{B}} (A \Diamond_b B)$

Embedding IL into LL

Definition (Girard 1987)

$$
\begin{array}{lcl}
(A_{\mathsf{at}})^\star & :\equiv & A_{\mathsf{at}} \\
(A \Diamond_z B)^\star & :\equiv & A^\star \Diamond_z B^\star \\
(A \to B)^\star & :\equiv & !A^\star \multimap B^\star \\
(\forall z A)^\star & :\equiv & \forall z A^\star \\
(\exists z A)^\star & :\equiv & \exists z A^\star .
\end{array}
$$

Lemma

If IL *proves* A *then* LL + (†) *proves* $A^\star$, *where*

$$(\dagger) \quad !\exists z A \multimap \exists z !A$$

Dialectica interpretation of LL

de Paiva (1989)

- Dialectica interpretation of LL
- interpretation of LL into CL in finite types
- focus on building a model of LL

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Shirahata (2006)

- relates de Paiva's interpretation to Shoenfield's classical variant of Gödel's interpretation

Outline

Dialectica interpretation

Definition (Gödel 1958)

$$
\begin{aligned}
(A \wedge B)_D(x, v; y, w) &:\equiv A_D(x; y) \wedge B_D(v; w) \\
(A \vee B)_D(x, v, z; y, w) &:\equiv A_D(x; y) \diamondsuit_z B_D(v; w) \\
(A \to B)_D(f, g; x, w) &:\equiv A_D(x; fwx) \to B_D(gx; w) \\
(\forall z A)_D(f; y, z) &:\equiv A_D(fz; y) \\
(\exists z A)_D(x, z; y) &:\equiv A_D(x; y).
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Then define $(A)^D :\equiv \exists x \forall y A_D(x; y)$.

Relational view

Interpretation assigns

- formulas A to binary relations $A_D(x; y)$
- proofs π of A to winning move t_π, i.e. $\forall y A_D(t_\pi; y)$

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I will write binary relation $A_D(x; y)$ as $|A|^x_y$.

Relational view

- Game for formula A is $(D_1, D_2, |A|^x_y \subseteq D_1 \times D_2)$
- **Two players**
  Eloise and Abelard
- **Two domains of moves**
  $x \in D_1$ and $y \in D_2$
- **Adjudication of winner**
  Relation $|A|^x_y$ between players' moves

Interpretation

$$|A \Diamond_z B|^{x,v}_{y,w} \;\; :\equiv \;\; |A|^x_y \Diamond_z |B|^v_w$$

Interpretation

$$|A \diamondsuit_z B|^{x,v}_{y,w} \quad :\equiv \quad |A|^x_y \diamondsuit_z |B|^v_w$$

$$|A \bindnasrepma B|^{f,g}_{y,w} \quad :\equiv \quad |A|^{fw}_y \bindnasrepma |B|^{gy}_w$$

$$|A \otimes B|^{x,v}_{f,g} \quad :\equiv \quad |A|^x_{fv} \otimes |B|^v_{gx}$$

Interpretation

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|A \otimes B|^{x,v}_{f,g} &:\equiv |A|^x_{fv} \otimes |B|^v_{gx} \\
|\exists z A(z)|^{x,z}_f &:\equiv |A(z)|^x_{fz} \\
|\forall z A(z)|^f_{y,z} &:\equiv |A(z)|^{fz}_y
\end{aligned}
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|?A|^f_y & :\equiv & ?|A|^{fy}_y \\
|!A|^x_f & :\equiv & !|A|^x_{fx}
\end{array}
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Consequences

Linear negation

$$|A^{\perp}|^{y}_{x} \equiv (|A|^{x}_{y})^{\perp}$$

Linear implication $(A \multimap B \equiv A^{\perp} \parr B)$

$$|A \multimap B|^{f,g}_{x,w} \equiv |A|^{x}_{gw} \multimap |B|^{fx}_{w}$$

Intuitionistic implication $(A \to B \equiv\ !A \multimap B)$

$$|A \to B|^{f,g}_{x,w} \equiv |A|^{x}_{gxw} \to |B|^{fx}_{w}$$

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Game $A^{\perp}$ is the same as A but with roles reversed

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Playing game B relative to game A

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Playing game B relative to (multiple copies of) game A

Soundness

Theorem

If LL $\vdash A$ *then* qfLL$^\omega$ *proves that Eloise has winning move in game* $|A|^x_y$.

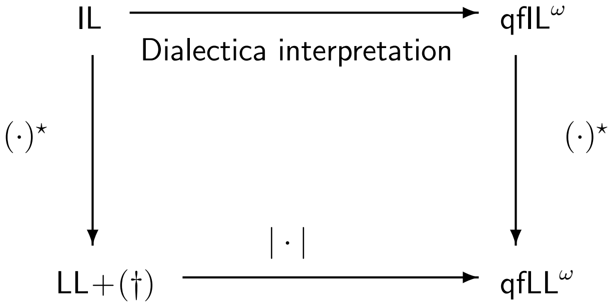
Theorem

If LL $\vdash A$ *then* qfLL$^\omega \vdash |A|^t_y$ *for some term* t.

Relation to Interpretation of IL

$$\text{IL} \xrightarrow[\text{Dialectica interpretation}]{} \text{qfIL}^{\omega}$$

Relation to Interpretation of IL



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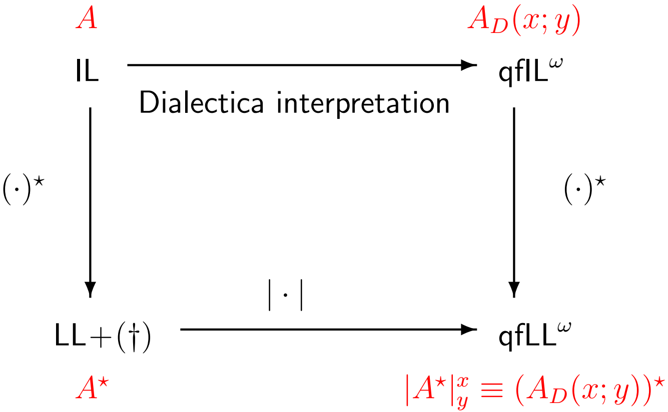

Characterisation

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- What about the other way around?
- For which extension of LL do we have the converse?

Characterisation

$$A$$

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$$A$$

$$\downarrow$$

$$\exists x \forall y |A|^x_y$$

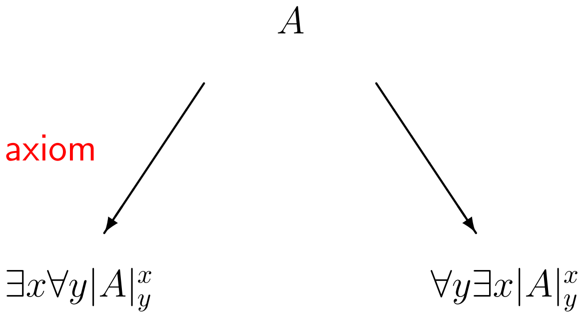
Characterisation

$$A$$

axiom

$$\exists x \forall y |A|^x_y$$

Characterisation



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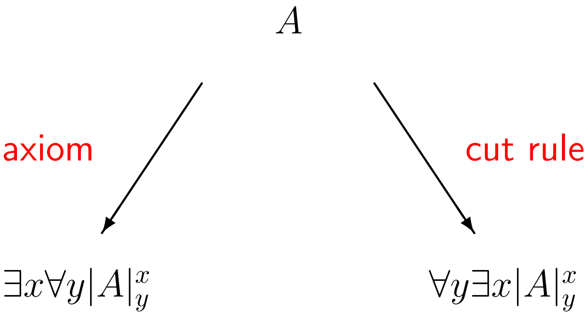

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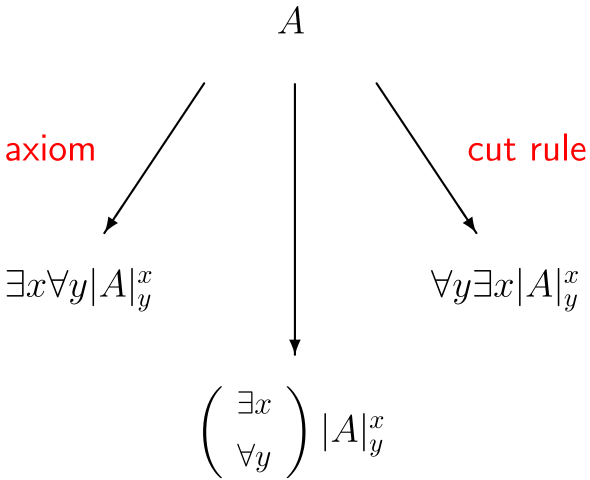

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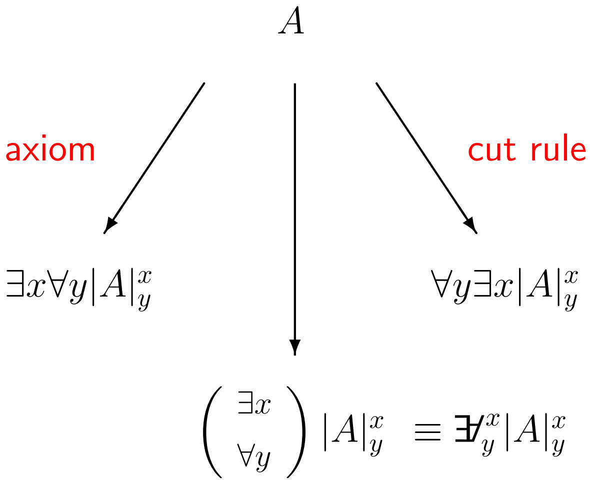

Simultaneous quantifier

$$\frac{A_0(a_0, y_0), \ldots, A_n(a_n, y_n)}{\exists\!\!\!/_{y_0}^{x_0} A_0(x_0, y_0), \ldots, \exists\!\!\!/_{y_n}^{x_n} A_n(x_n, y_n)}$$

(∗) y_i may only appear free in the terms a_j, for $j \neq i$;

New Principles

- **Sequential choice**

  $\forall z \exists\!\!\!\forall^x_y A(x, y, z) \multimap \exists\!\!\!\forall^f_{y,z} A(fz, y, z)$

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  $$\exists\!\!\forall^x_y A(x) \parr \exists\!\!\forall^v_w B(v) \multimap \exists\!\!\forall^{f,g}_{y,w} (A(fw) \parr B(gy))$$

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- **Markov principle**

  $\forall x ! A_{\mathsf{qf}} \multimap ! \forall x A_{\mathsf{qf}}$

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  $\exists\!\!\!\forall^x_y A(x) \otimes \exists\!\!\!\forall^v_w B(v) \multimap \exists\!\!\!\forall^{f,g}_{y,w} (A(fw) \otimes B(gy))$
- **Markov principle**
  $\forall x ! A_{\mathsf{qf}} \multimap ! \forall x A_{\mathsf{qf}}$
- **Trump advantage**
  $! \exists\!\!\!\forall^x_y A \multimap \exists x ! \forall y A$

New Principles

Theorem

These principles are necessary and sufficient for deriving the equivalence between A *and its interpretation* $\exists\!\!\!/_y^x |A|_y^x$.

Outline

Unifying Functional Interpretations

- **Dialectica**

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- **Diller-Nahm**

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Unifying Functional Interpretations

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- **Modified realizability**

$$|!A|^x \quad :\equiv \quad !\forall y |A|^x_y$$

$$|?A|_y \quad :\equiv \quad ?\exists x |A|^x_y$$

Hybrid interpretations

- Linear logic modalities are not canonical
- Make use of multi-modal LL $(?_g, ?_d, ?_k)$ with

$$\frac{\Gamma, ?_g A}{\Gamma, ?_d A} \qquad \frac{\Gamma, ?_d A}{\Gamma, ?_k A}$$

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- Hybrid interpretation

  Possible to deal with proofs involving both MP *and* EXT

$$\text{MP} \quad : \quad ?_g \exists x A_{\mathsf{qf}}(x) \to \exists x ?_g A_{\mathsf{qf}}(x)$$

$$\text{EXT} \quad : \quad !_k \forall n (\alpha n = \beta n) \to Y\alpha = Y\beta$$

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Usual Dialectica

Definition (Dialectica Interpretation)

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(A \wedge B)_D(x, v; y, w) &:\equiv A_D(x; y) \wedge B_D(v; w)\\
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\begin{array}{lcl}
(A \wedge B)_D(x, v; y, w, z) & :\equiv & A_D(x; y) \lozenge_z B_D(v; w) \\
(A \vee B)_D(x, v, z; y, w) & :\equiv & A_D(x; y) \lozenge_z B_D(v; w) \\
(A \to B)_D(f, g; x, w) & :\equiv & A_D(x; fwx) \to B_D(gx; w) \\
(\forall z A)_D(f; y, z) & :\equiv & A_D(fz; y) \\
(\exists z A)_D(x, z; f) & :\equiv & A_D(x; fz).
\end{array}
$$

Then define $(A)^D :\equiv \exists x \forall y A_D(x; y)$.

Reformulation of Dialectica

Even with this reformulation we have:

Theorem

If

$$\mathsf{IL}^{\omega} \vdash A$$

then

$$\mathsf{IL}^{\omega} \vdash \forall y A_D(t, y)$$

for some term t.