Computational Interpretations of Classical Linear Logic

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Outline

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Proof Interpretations

Syntax	$\Rightarrow$	**Semantics**
e.g. "blue car"		real blue cars

Proof Interpretations

$$\left\{ \begin{array}{l} \textbf{Syntax} \\ \text{e.g. "blue car"} \end{array} \right. \Rightarrow \left. \begin{array}{r} \textbf{Semantics} \\ \text{real blue cars} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \textbf{Syntax} \\ \text{e.g. Java} \end{array} \right. \Rightarrow \left. \begin{array}{r} \textbf{Syntax} \\ \text{Machine code} \end{array} \right\}$$

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$$\left\{ \begin{array}{lcr} \textbf{Proofs} & & \textbf{Proofs} \\ \text{e.g. CL} & \Rightarrow & \text{IL} \\ \text{e.g. PA} & & \text{PRA} \end{array} \right\}$$

Functional Interpretations

$$\textbf{Formula } A \quad \Rightarrow \quad \textbf{Set of functionals } |A|$$

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$S_1 \vdash_\pi A$ $\Rightarrow$ $S_2 \vdash f_\pi \in |A|$

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Formula A	$\Rightarrow$	**Set of functionals** $\|A\|$
$\forall x \exists y (y > x)$		$\{f \; : \; fx > x\}$
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$$S_1 \vdash_\pi A \quad \Rightarrow \quad S_2 \vdash f_\pi \in |A|$$

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$\ldots$		$\lambda x . x + 1$
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Functional Interpretations

Dialectica

- Gödel 1958
- Relative consistency of arithmetic

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$$\begin{cases} \text{Falsity interpreted as empty set } (|\perp| \equiv \emptyset) \\ \mathsf{PA} \vdash \perp \quad \Rightarrow \quad \mathsf{PRA}^\omega \vdash \exists f (f \in \emptyset) \end{cases}$$

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Modified realizability

- Kreisel 1959
- Independence results for IL

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Modified realizability

- Kreisel 1959
- Independence results for IL

$$\begin{cases} |P| \text{ set of non-computable functionals} \\ \mathsf{IL} \vdash P \quad \Rightarrow \quad \mathsf{IL} \vdash \exists f (f \in |P|) \end{cases}$$

Functional Interpretations

- Gödel's Dialectica interpretation
- Kreisel's modified realizability
- Diller-Nahm interpretation
- Stein's family of interpretations
- Monotone variants of the above (Kohlenbach)
- Bounded Dialectica interpretation (Ferreira/O.)
- Bounded modified realizability (Ferreira)
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Goal

Understand different functional interpretations

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Functional interpretation of a refinement of IL

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Understand different functional interpretations

Functional interpretation of a refinement of IL

Linear logic

Linear Logic (Girard 1987)

- Explicit treatment of **contraction**

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \quad \Rightarrow \quad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$$

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- **Refinement** of intuitionistic implication

$$A \to B \quad \equiv \quad !A \multimap B$$

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- **Refinement** of intuitionistic implication

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- **Refinement** of logical connectives

	conjunction	disjunction
additive	$\wedge$	$\vee$
multiplicative	$\otimes$	$\invamp$

Linear Logic: Duality

$$(A \vee B)^{\perp} \equiv A^{\perp} \wedge B^{\perp} \qquad (\exists z A)^{\perp} \equiv \forall z A^{\perp}$$

$$(A \wedge B)^{\perp} \equiv A^{\perp} \vee B^{\perp} \qquad (\forall z A)^{\perp} \equiv \exists z A^{\perp}$$

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$$A \multimap B \equiv A^\perp \parr B$$

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Linear Logic: Structural

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B} \text{(cut)} \qquad A \vdash A \quad \text{(id)}$$

$$\frac{\Gamma, A \vdash B}{\Gamma, B^{\perp} \vdash A^{\perp}} (\perp) \qquad \frac{\Gamma \vdash A}{\pi\{\Gamma\} \vdash A} \text{(per)}$$

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Linear Logic: Connectives and Quantifiers

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall z A} \qquad \frac{\Gamma \vdash A[t/z]}{\Gamma \vdash \exists z A}$$

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Linear Logic: Modalities

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \text{(con)} \qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \text{(wkn)}$$

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash ?A}\ (?)$$

Outline

Mathematicians are happy with proof or counter-example

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Mathematics is like a game,
mathematicians are always winners
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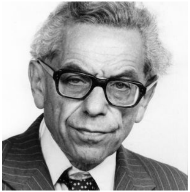
View a mathematical statement as the description of a game

$$\forall n \geq 2\, \exists x, y, z (\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z})$$



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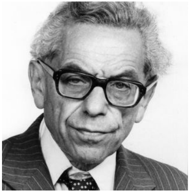

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$$f_0, f_1, f_2 : \mathbb{N} \to \mathbb{N}^* \qquad n \in \{2, \ldots\}$$

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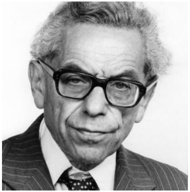

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Games: Formal Description

- Game $G \equiv (D_1, D_2, R \subseteq D_1 \times D_2)$

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  Eloise and Abelard

- **Two domains of moves**

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- **Adjudication of Winner**

  Relation $R(x, y)$ between players' moves

  (*usually* $|G|^x_y$)

Games: Examples

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$x \in \mathbb{N}$	$y \in \mathbb{N}$	$x \geq y$
$f \in \mathbb{N} \to \mathbb{N}$	$y \in \mathbb{N}$	$f(y) \geq y$

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$f \in \mathbb{N} \to \mathbb{N}$	$y \in \mathbb{N}$	$f(y) \geq y$
$f_i \in \mathbb{N} \to \mathbb{N}^*$	$n \geq 2$	$\frac{4}{n} = \frac{1}{f_0 n} + \frac{1}{f_1 n} + \frac{1}{f_2 n}$

Goal

A is true (is provable)

iff

Eloise has winning move in game $|A|^x_y$

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Symmetry

Game $A^{\perp}$ should be game A with roles reversed

$$|A^{\perp}|^{x}_{y} \equiv \neg |A|^{y}_{x}$$

$$|(A^{\perp})^{\perp}|^{x}_{y} \equiv |A|^{x}_{y}$$

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$$|A \otimes B|^{x,v}_{f,g} \quad :\equiv \quad |A|^{x}_{fv} \text{ and } |B|^{v}_{gx}$$

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$$|\forall z A(z)|^{f}_{y,z} \quad :\equiv \quad |A(z)|^{fz}_{y}$$

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Exponential Games

One of the players plays first (**break symmetry**)

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(2) finitely many copies

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(3) no copying allowed (make single move)

$$\begin{cases} |!A|^x_f & :\equiv \quad |A|^x_{fx} \\ |?A|^f_y & :\equiv \quad |A|^{fy}_y \end{cases}$$

Soundness

Theorem

If

$\Gamma \vdash A$ *is provable in linear logic*

then

Eloise wins game $\Gamma \multimap A$

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Theorem

If

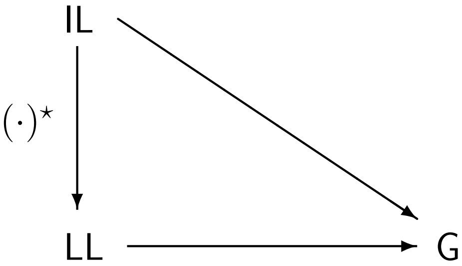
$\Gamma \vdash A$ *is provable in linear logic*

then

Eloise wins game $\Gamma \multimap A$ *, i.e.*

she has moves t, r *such that for all* v, y

$$|\Gamma|^{v}_{r(y)} \vdash |A|^{t(v)}_{y}$$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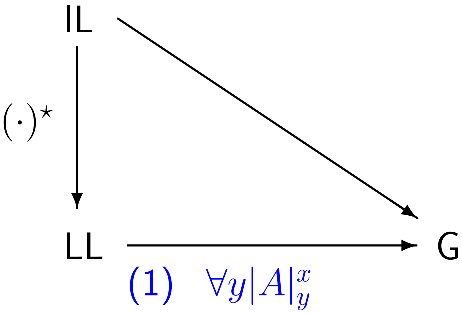
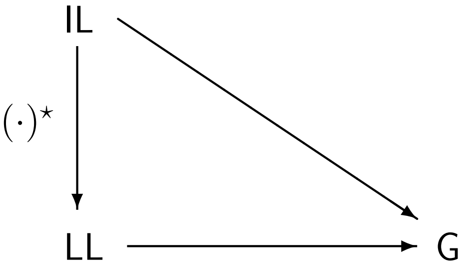
Modified realizability (Kreisel'1959)



Diller-Nahm interpretation (Diller-Nahm'1974)

Modified realizability (Kreisel'1959)



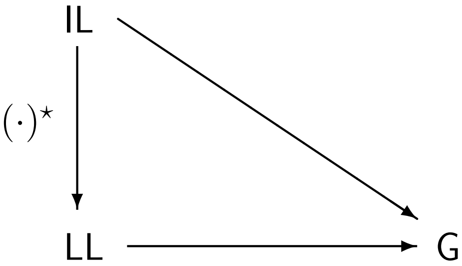
(1) $\forall y |A|^x_y$

(2) $\forall y \in f x \, |A|^x_y$

Dialectica interpretation (Gödel'1958)

Diller-Nahm interpretation (Diller-Nahm'1974)

Modified realizability (Kreisel'1959)



(1) $\forall y |A|^x_y$

(2) $\forall y \in f x \, |A|^x_y$

(3) $|A|^x_{fx}$

Parametrised Interpretation

Bounded quantifiers $\forall x^{\rho} \sqsubset a^{\rho^*} A$ and $\exists x^{\rho} \sqsubset a^{\rho^*} A$

$$(\forall x \sqsubset a\, A)^{\perp} \equiv \exists x \sqsubset a\, A^{\perp}$$
$$(\exists x \sqsubset a\, A)^{\perp} \equiv \forall x \sqsubset a\, A^{\perp}$$

Parametrised interpretation:

$$|!A|^{x}_{f} :\equiv \forall y \sqsubset f x\, |A|^{x}_{y}$$
$$|?A|^{f}_{y} :\equiv \exists x \sqsubset f y\, |A|^{x}_{y}$$

Parametrised Interpretation

$\dfrac{\Gamma \vdash B}{\Gamma, !A \vdash B}$	$c_A \ : \ \rho^*$
$\dfrac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$	$\varepsilon_A \ : \ \rho^* \times \rho^* \hookrightarrow \rho^*$
$\dfrac{\Gamma \vdash A}{\Gamma \vdash ?A}$	$\eta_A \ : \ \rho \hookrightarrow \rho^*$
$\dfrac{!\Gamma \vdash A}{!\Gamma \vdash !A}$	$\mu_A \ : \ \rho \to \tau^* \hookrightarrow \rho^* \to \tau^*$

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Characterisation

- We have seen that if A is provable in LL then Eloise has a winning move

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- What about the other way around?

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- What about the other way around?
- For which extension of LL do we have the converse?

Characterisation (modified realizability)

$$
\begin{array}{lcl}
|A \otimes B|^{x,v}_{f,g} & :\equiv & |A|^{x}_{fv} \otimes |B|^{v}_{gx} \\
|A \multimap B|^{f,g}_{x,w} & :\equiv & |A|^{x}_{fw} \multimap |B|^{gx}_{w} \\
|\forall z A(z)|^{f}_{y,z} & :\equiv & |A(z)|^{fz}_{y} \\
|\exists z A(z)|^{x,z}_{f} & :\equiv & |A(z)|^{x}_{fz} \\
|!A|^{x} & :\equiv & !\forall y |A|^{x}_{y} \\
|?A|_{y} & :\equiv & ?\exists x |A|^{x}_{y}
\end{array}
$$

Characterisation (modified realizability)

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axiom

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$$\forall w \exists v |\Gamma|^{v}_{w} \vdash \exists x \forall y |A|^{x}_{y}$$

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cut rule

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$$\forall w \exists v |\Gamma|^{v}_{w} \vdash \exists x \forall y |A|^{x}_{y}$$

$$\exists v \forall w |\Gamma|^{v}_{w} \vdash \forall y \exists x |A|^{x}_{y}$$

$$\exists\!\!\!\forall^{v}_{w} |\Gamma|^{v}_{w} \vdash \exists\!\!\!\forall^{x}_{y} |A|^{x}_{y}$$

Simultaneous Quantifier

$$\frac{A_0(a_0, y_0), \ldots, A_n(a_n, y_n)}{\exists\!\!\!/^{x_0}_{y_0} A_0(x_0, y_0), \ldots, \exists\!\!\!/^{x_n}_{y_n} A_n(x_n, y_n)} \; (\exists\!\!\!/)$$

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$$\frac{(x = y), (x \neq y)}{\exists\!\!\!/^{x}_{y} (x = y), \exists\!\!\!/^{y}_{x} (x \neq y)} \; (\exists\!\!\!/)$$

New Principles

- Sequential choice

  $\forall z \exists\!\!\!\forall^x_y A(x, y, z) \multimap \exists\!\!\!\forall^f_{y,z} A(fz, y, z)$

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  $$\forall z \exists\!\!\forall^x_y A(x, y, z) \multimap \exists\!\!\forall^f_{y,z} A(fz, y, z)$$

- Parallel choice

  $$\exists\!\!\forall^x_y A(x) \otimes \exists\!\!\forall^v_w B(v) \multimap \exists\!\!\forall^{f,g}_{y,w} (A(fw) \otimes B(gy))$$

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Theorem

These are sufficient for deriving the equivalence between A and its interpretation $\exists\!\!\!/_{y}^{x} |A|_{y}^{x}$.

Summary

- Functional interpretations of linear logic

  Usual interpretations of IL derivable

- Interesting use of (simple) branching quantifier

  Characterisation using branching quantifier

- Sound extensions of linear logic